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# MOMENTUM AND ENERGY EQUATIONS FOR DISPERSE TWO-PHASE FLOWS AND THEIR CLOSURE FOR DILUTE SUSPENSIONS

D. Z. ZHANG† and A. PROSPERETTI
Department of Mechanical Engineering, The Johns Hopkins University, Baltimore, MD 21218, U.S.A.

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**Abstract**—The averaged momentum and energy equations for disperse two phase flows are derived by extending a recently developed ensemble averaging method. The resulting equations have a 'two-fluid' form and the closure problem is phrased in terms of quantities that are amenable to direct numerical simulation. An application of the general theory is given in the dilute limit (first-order in the particle volume fraction), and at low particle Reynolds number. In this case, an analytical closure of the equations free of *ad hoc* approximations is explicitly given. New effects due to non-uniform particle distribution are identified. © 1997 Elsevier Science Ltd. All rights reserved.

*Key Words:* disperse flows, averaged equations, ensemble averaging

## 1. INTRODUCTION

In earlier papers (Zhang and Prosperetti 1994a, 1994b) we have presented an approach to the derivation of averaged equations for disperse two-phase flows. Those papers dealt with inviscid suspensions of spherical particles and bubbles respectively. The purpose of this paper is to present a more systematic version of the averaging technique, to apply it to viscous flow, and to derive the average form of the energy equations of the two phases. The results are illustrated with applications to slow, conduction-dominated, viscous flow.

Our method is based on ensemble averaging but differs from earlier work in several respects. In the first place, we use phase ensemble averaging for the continuous phase (i.e. we average over all the configurations such that at time $t$ the position $\mathbf{x}$ is occupied by the continuous phase), but not for the disperse phase. For the latter we introduce a 'particle' ensemble average in which global particle attributes (e.g. the velocity of the center of mass) are averaged directly. A similar idea can be found in Anderson and Jackson (1967) and, more recently, in Jackson (1996). These papers are however based on volume averaging. The advantage of particle averaging is the ability to limit the number of degrees of freedom used in the description of the disperse phase tailoring it to the specific situation at hand. For example, for rigid particles, one can account for the constraint of rigidity directly by describing each particle in terms of its linear and angular velocity.

Secondly, by explicitly and systematically using the 'small-particle approximation' (Zhang and Prosperetti 1994a), i.e. the assumption that the particle size $a$ is small compared with the macroscopic characteristic length $L$, we can considerably simplify the derivation of the averaged equations. Thirdly, unlike some of the earlier work (e.g. Hinch 1977), we derive equations of the so-called 'two-fluid' form widely used in Engineering (see, e.g. Drew 1983; Wallis 1991).

The present approach seems to offer several major advantages on the available ones:

(1) It is very flexible and can be applied unchanged to a wide variety of situations. We have dealt with potential-flow problems in the two earlier papers already referred to. Here we study the viscous problem and convective–diffusive heat transfer by identical techniques. This feature is particularly useful in the dilute limit where explicit results can be obtained systematically in a unified way. At

†*Present address*: Theoretical Division, Fluid Dynamics Group, Los Alamos National Laboratory, Los Alamos, NM 87545, U.S.A.

the next higher order in the disperse-phase volume fraction, the equations of Hinch (1977) are recovered (Zhang 1997).

(2) The closure problem presents itself in a form that can be effectively tackled by means of direct numerical simulation. We have given an example of this approach for the linear potential problem in Zhang and Prosperetti (1994a). Work for the nonlinear case and for Stokes flow is under way.

(3) The method is also applicable to situations in which the mixture is not spatially uniform provided the ratio of the largest particle dimension to the scale of variation of the average quantities is smaller than $O(1)$. This circumstance enables us to throw some new light on available results, e.g. for the so-called 'particle stress' and point out their limitations (section 8).

Our methods are applicable to suspensions of solids as well as droplets. In the interest of simplicity of exposition, only the solid case is considered in the text. Droplets are considered in appendix B. A general description of the approach including details omitted here in the interest of brevity can be found in an expository paper by one of the authors (Prosperetti 1997).

## 2. AVERAGING AND CONTINUITY EQUATIONS

In this section we introduce and extend some definitions and results of the ensemble phase averaging method that will be needed below. The reader is referred to our earlier papers for details (Zhang and Prosperetti 1994a,b).

We consider an ensemble of macroscopically identical suspensions of $N$ spherical particles in a fluid continuous phase. We use the word *configuration* and the symbol $\mathscr{C}^N$ to indicate the values of a number of degrees of freedom sufficient to specify uniquely the dynamical state of the system at time $t$. In particular, $\mathscr{C}^N$ will include the current values of the particles' degrees of freedom such as position of the center $\{\mathbf{y}^\alpha\}$, $\alpha = 1, 2, \ldots, N$, center-of-mass velocity $\{\mathbf{w}^\alpha\}$, and others.†

Let $P(N; t)$ be the probability density of encountering the configuration $\mathscr{C}^N$ in the ensemble of flows at time $t$. Since we assume the particles to be indistinguishable, it is convenient to use the normalization

$$\int \mathrm{d}\mathscr{C}^N P(N; t) = N! \qquad [2.1]$$

where $\mathrm{d}\mathscr{C}^N$ is the volume element in the phase space of the sytem.

Let $\chi_{C,D}(\mathbf{x}; N)$ be the characteristic, or indicator, functions for the continuous and disperse phases, respectively. For a suspension of equal spherical particles of radius $a$ an explicit representation is (Lundgren 1972)

$$\chi_D(\mathbf{x}; N) = 1 - \chi_C(\mathbf{x}; N) = \sum_{\alpha=1}^{N} H(a - |\mathbf{x} - \mathbf{y}^\alpha|), \qquad [2.2]$$

where $H$ is the Heaviside distribution. In terms of these characteristic functions the volume fractions $\epsilon_{C,D}$ of the phases are defined by

$$\epsilon_{C,D}(\mathbf{x}, t) = \frac{1}{N!} \int \mathrm{d}\mathscr{C}^N P(N; t) \chi_{C,D}(\mathbf{x}; N). \qquad [2.3]$$

By using the representation [2.2] it is easy to show that, as pointed out by Lundgren (1972),

$$\epsilon_D(\mathbf{x}, t) = \int_{|\mathbf{x}-\mathbf{y}| \leq a} \mathrm{d}^3 y \int \mathrm{d}^3 w P(\mathbf{y}, \mathbf{w}; t), \qquad [2.4]$$

†The continuous-phase degrees of freedom are entirely dependent on those of the particles only for potential and Stokes flow. At finite Reynolds numbers, they must be explicitly included in $\mathscr{C}^N$ to specify a configuration uniquely.

where

$$P(\mathbf{y}, \mathbf{w}; t) \equiv P(1; t) = \frac{1}{(N-1)!} \int \mathrm{d}\mathscr{C}^{N-1} P(N; t), \tag{2.5}$$

is the reduced one-particle distribution function. In [2.5] and in the following we write $\int \mathrm{d}\mathscr{C}^{N-1}$ to indicate integration over all the degrees of freedom of the system except the position and velocity of particle 1.

Upon expanding $P(1; t)$ in Taylor series around $\mathbf{x}$ we find

$$\epsilon_{\mathrm{D}} = nv + \tfrac{1}{10} v a^2 \nabla^2 n + o\left(\frac{a^2}{L^2} nv\right), \tag{2.6}$$

where $L$ is the scale of variation of the averaged quantities, $v = \frac{4}{3}\pi a^3$ is the particle volume, and $n$ is the particle number density defined by

$$n(\mathbf{x}, t) = \int \mathrm{d}^3 w P(\mathbf{x}, \mathbf{w}; t). \tag{2.7}$$

For a field quantity $f_{\mathrm{C,D}}(\mathbf{x}, t; N)$ pertaining to the continuous or disperse phase, the phase averages are defined by averaging over all the configurations in which the point $\mathbf{x}$ is in the appropriate phase:

$$\langle f_{\mathrm{C,D}} \rangle(\mathbf{x}, t) = \frac{1}{N! \epsilon_{\mathrm{C,D}}} \int f_{\mathrm{C,D}}(\mathbf{x}, t; N) \chi_{\mathrm{C,D}}(\mathbf{x}; N) P(N; t)\, \mathrm{d}\mathscr{C}^N. \tag{2.8}$$

A consequence of this definition is that differentiation and averaging do not commute. For example

$$\nabla \langle f_{\mathrm{C}} \rangle = \langle \nabla f_{\mathrm{C}} \rangle + \frac{1}{\epsilon_{\mathrm{C}}} \int \mathrm{d}^3 w \int_{|\mathbf{x}-\mathbf{y}|=a} \mathrm{d}S_{\mathbf{y}} P(\mathbf{y}, \mathbf{w}; t)[\langle f_{\mathrm{C}} \rangle_1(\mathbf{x}, t \mid \mathbf{y}, \mathbf{w}) - \langle f_{\mathrm{C}} \rangle(\mathbf{x}, t)]\mathbf{n}, \tag{2.9}$$

where $\mathbf{n}$ is the outward unit normal on the particle surface. Here $\langle f_{\mathrm{C}} \rangle_1(\mathbf{x}, t \mid \mathbf{y}, \mathbf{w})$ is the one-particle conditional average of $f_{\mathrm{C}}$, i.e. the average taken only over the subset of the ensemble such that, at time $t$, one particle with velocity $\mathbf{w}$ is centered at $\mathbf{y}$:

$$\langle f_{\mathrm{C,D}} \rangle_1(\mathbf{x}, t \mid \mathbf{y}, \mathbf{w}) = \frac{1}{(N-1)! \epsilon^1_{\mathrm{C,D}}} \int \mathrm{d}\mathscr{C}^{N-1} \chi_{\mathrm{C,D}}(\mathbf{x}; N) f_{\mathrm{C,D}}(\mathbf{x}, t; N) P(N-1 \mid \mathbf{y}, \mathbf{w}; t). \tag{2.10}$$

Here, $P(N-1 \mid \mathbf{y}, \mathbf{w}; t) = P(N; t)/P(\mathbf{y}, \mathbf{w}; t)$ is the conditional probability and $\epsilon^1_{\mathrm{C,D}}$ are the one-particle conditional volume fractions.

As it stands, the integral in [2.9] is inconvenient to evaluate as integration is over all the configurations in which there is a particle touching the point $\mathbf{x}$ rather than over a particle surface. To circumvent this difficulty, it is useful to adopt the following procedure, that we present in general terms as later it will have to be applied to several different quantities. Let $\mathbf{r} = \mathbf{x} - \mathbf{y}$ and define

$$F(\mathbf{r}, \mathbf{y}, t) \equiv P(\mathbf{y}, \mathbf{w}; t)[\langle f_{\mathrm{C}} \rangle_1(\mathbf{y} + \mathbf{r}, t \mid \mathbf{y}, \mathbf{w}) - \langle f_{\mathrm{C}} \rangle(\mathbf{y} + \mathbf{r}, t)]. \tag{2.11}$$

In the integral appearing in [2.9] $r = a$. In many situations of interest $F$ may be expected to depend only weakly on the position $\mathbf{y}$ of the particle center so that one may use Taylor's theorem centered around $\mathbf{x}$ and write

$$F(\mathbf{r}, \mathbf{y}, t) = F(\mathbf{r}, \mathbf{x}, t) - \mathbf{r} \cdot \nabla F(\mathbf{r}, \mathbf{x} - \mathbf{h}, t), \tag{2.12}$$

where $|\mathbf{h}| < a$ and the Lagrange form of the remainder has been used. (In this relation, $\mathbf{r}$ must be interpreted as the distance from the particle center that has moved from the position $\mathbf{y}$ to $\mathbf{x}$). The idea underlying [2.12] can be further clarified with reference to figure 1. The quantity $F$ defined

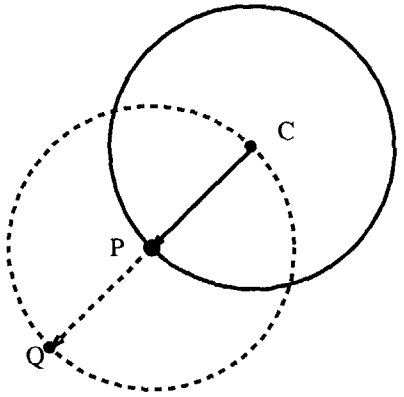


Figure 1. Illustration of the approximation used in deriving [2.13]. According to [2.9], the value of the integrand at the point P for all the particles touching that point is needed. The solid line shows one such particle centered at C. The contribution of this particle is approximated by the integrand at Q, plus a few terms of the Taylor series expansion. In other words, the contribution of the particle centered at C is approximated by that centered at P.

in [2.11], as it appears in the integral [2.9], should be evaluated at the point P. Equation [2.12] expresses it as its value at the neighboring point Q, plus a correction.

Upon substituting [2.12] into [2.9] we then find

$$\epsilon_C \nabla \langle f_C \rangle = \epsilon_C \langle \nabla f_C \rangle + \epsilon_D \mathscr{A}[f_C] - \nabla \cdot (\epsilon_D \mathscr{L}[f_C]), \tag{2.13}$$

where

$$\epsilon_D \mathscr{A}[f_C] = \int d^3w P(\mathbf{x}, \mathbf{w}; t) \int_{|\mathbf{x}-\mathbf{z}|=a} dS_z [\langle f_C \rangle_1(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w}) - \langle f_C \rangle(\mathbf{z}, t)]\mathbf{n}, \tag{2.14}$$

$$\epsilon_D \mathscr{L}[f_C] = a \int d^3w \int_{|\mathbf{x}-\mathbf{z}|=a} dS_z P(\mathbf{x}-\mathbf{h}, \mathbf{w}; t)[\langle f_C \rangle_1(\mathbf{z}-\mathbf{h}, t \mid \mathbf{x}-\mathbf{h}, \mathbf{w}) - \langle f_C \rangle(\mathbf{z}-\mathbf{h}, t)]\mathbf{nn}, \tag{2.15}$$

with $\mathbf{h}$ a function of $\mathbf{z}$. Note that, unlike [2.9], the integral in the definition [2.14] of $\mathscr{A}[f_C]$ is taken over the surface of a single particle centered at $\mathbf{x}$. Further, it is evident that $\mathscr{L}[f_C]$ is non-zero only insofar as the flow in the neighborhood of a particle differs from the mean flow. It is, therefore, clear that this term will account for the microphysics of the suspension.

For particles small with respect to the scale $L$ of macroscopic variation of the flow, $\mathscr{L}[f_C]$ can be expanded retaining only the first few terms to find (cf. Zhang and Prosperetti 1994a)

$$\epsilon_D \mathscr{L}[f_C] = \epsilon_D \mathscr{T}[f_C] + \nabla \cdot (\epsilon_D \mathscr{S}[f_C]) + \nabla\nabla : (\epsilon_D \mathscr{R}[f_C]) + O\left(\frac{a^3}{L^3} \epsilon_D \langle f_C \rangle_1\right), \tag{2.16}$$

where

$$\epsilon_D \mathscr{T}[f_C] = a \int d^3w P(\mathbf{x}, \mathbf{w}; t) \int_{|\mathbf{x}-\mathbf{z}|=a} dS_z [\langle f_C \rangle_1(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w}) - \langle f_C \rangle(\mathbf{z}, t)]\mathbf{nn}, \tag{2.17}$$

$$\epsilon_D \mathscr{S}[f_C] = -\tfrac{1}{2} a^2 \int d^3w P(\mathbf{x}, \mathbf{w}; t) \int_{|\mathbf{x}-\mathbf{z}|=a} dS_z [\langle f_C \rangle_1(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w}) - \langle f_C \rangle(\mathbf{z}, t)]\mathbf{nnn}, \tag{2.18}$$

$$\epsilon_D \mathscr{R}[f_C] = \tfrac{1}{6} a^3 \int d^3w P(\mathbf{x}, \mathbf{w}; t) \int_{|\mathbf{x}-\mathbf{z}|=a} dS_z [\langle f_C \rangle_1(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w}) - \langle f_C \rangle(\mathbf{z}, t)]\mathbf{nnnn}. \tag{2.19}$$

Clearly, the part of $\mathscr{S}$ isotropic in the last two indices is proportional to $\mathscr{A}$, etc. While these properties may be useful to simplify the calculation of the integrals, they are of no particular value for the purposes of the present paper and their analysis will not be pursued.

Contrary to [2.9], all these integrals are effected over the surface of a particle centered at $\mathbf{x}$. This circumstance introduces a significant simplification, some aspects of which will be seen below. Furthermore, in numerical simulations, it is next to impossible to evaluate the original integral [2.9], while [2.14], [2.17]–[2.19] are much easier to deal with. It should be noted that an important situation where the expansion [2.16] is invalid is the vicinity of boundaries. Here the integral in [2.9] cannot be approximated.

In our earlier studies on particles in inviscid fluids (Zhang and Prosperetti 1994a, 1994b) it was only necessary to carry the first term of the expansion of $\mathscr{L}[f_C]$. It will be seen below, however, that additional terms are necessary for consistency in the case of the viscous stress tensor and of the heat flux.

A relation similar to [2.9] holds for the time derivative, but for the present purposes we only need to state the following transport theorem†

$$\frac{\partial \epsilon_C \langle f_C \rangle}{\partial t} + \nabla \cdot (\epsilon_C \langle f_C \mathbf{u}_C \rangle) = \epsilon_C \left\langle \frac{\partial f_C}{\partial t} + \nabla \cdot (f_C \mathbf{u}_C) \right\rangle, \tag{2.20}$$

where $\mathbf{u}_C$ is the velocity of the continuous phase. The time derivative in the right-hand side is taken at constant $\mathbf{x}$, allowing the particles to move. In particular, by setting here $f_C = 1$, we find the continuity equation for the continuous phase, assumed to be incompressible:

$$\frac{\partial \epsilon_C}{\partial t} + \nabla \cdot (\epsilon_C \langle \mathbf{u}_C \rangle) = 0. \tag{2.21}$$

In dealing with the disperse phase, a different kind of average proves much more useful than [2.8]. Let $g^{(1)}(N; t)$ be a single-particle quantity that can be considered as belonging to the particle as a whole. Examples are the center-of-mass velocity and acceleration, the angular velocity, and many others. Then we define the average of $g^{(1)}$ by

$$\bar{g}(\mathbf{x}, t) = \frac{1}{n(\mathbf{x}, t)(N-1)!} \int \mathrm{d}^3 w \int \mathrm{d}\mathscr{C}^{N-1} P(N; t) g^{(1)}(N; t). \tag{2.22}$$

If $g^{(1)}$ does not depend explicitly on the configuration of the other particles, the definition [2.22] reduces to

$$\bar{g}(\mathbf{x}, t) = \frac{1}{n(\mathbf{x}, t)} \int \mathrm{d}^3 w P(\mathbf{x}, \mathbf{w}; t) g^{(1)}(\mathbf{x}, \mathbf{w}; t). \tag{2.23}$$

A description in terms of such 'particle-averaged' quantities is particularly attractive whenever a resolution of the fine degrees of freedom of the particles, e.g. internal stress or detailed temperature distribution, is undesirable or unnecessary (see, however, the comments at the end of appendix A). This situation of course arises very frequently in multiphase flow.

It can be proven that $\bar{g}$ satisfies the following transport theorem (Zhang and Prosperetti 1994a)

$$\frac{\partial n \bar{g}}{\partial t} + \nabla \cdot (n \overline{g \mathbf{w}}) = n \overline{\frac{\mathrm{d}g}{\mathrm{d}t}}, \tag{2.24}$$

in which the time derivative in the right-hand side is taken following the motion of all the particles.

By setting $g^{(1)} = 1$ in [2.24] we find an equation for the conservation of the particle number density

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \bar{\mathbf{w}}) = 0, \tag{2.25}$$

†In Zhang and Prosperetti (1994a) this relation was derived for the case of potential flow in which the probability space is smaller. However, it is easy to convince oneself that the same relation holds in general.

where, according to [2.23],

$$\bar{\mathbf{w}}(\mathbf{x}, t) = \frac{1}{n(\mathbf{x}, t)} \int \mathrm{d}^3 w P(\mathbf{x}, \mathbf{w}; t)\mathbf{w}. \tag{2.26}$$

The velocity field defined by this relation is different from the phase-average disperse-phase velocity obtained from the definition [2.8] which, as is easily shown by using [2.2], can be reduced to

$$\epsilon_D \langle \mathbf{u}_D \rangle(\mathbf{x}, t) = \int \mathrm{d}^3 w \int_{|\mathbf{y}-\mathbf{x}| \leqslant a} \mathrm{d}^3 y P(\mathbf{y}, \mathbf{w}; t) \langle \mathbf{u}_D \rangle_1(\mathbf{x}, t \mid \mathbf{y}, \mathbf{w})$$

$$= nv\bar{\mathbf{w}} + \int \mathrm{d}^3 w \int_{|\mathbf{y}-\mathbf{x}| \leqslant a} \mathrm{d}^3 y [P(\mathbf{y}, \mathbf{w}; t) \langle \mathbf{u}_D \rangle_1(\mathbf{x}, t \mid \mathbf{y}, \mathbf{w}) - P(\mathbf{x}, \mathbf{w}; t)\mathbf{w}]. \tag{2.27}$$

Conceptually, the difference between $nv\bar{\mathbf{w}}$ and $\epsilon_D \langle \mathbf{u}_D \rangle$ is that the former quantity gives the disperse-phase volume flux due to all the particles *whose center is inside* a unit (macroscopic) volume, while the latter only counts the disperse phase material that lies *entirely* within the unit volume. Unless the significant macroscopic length scale $L$ is very small, therefore, the difference between these two quantities may be expected to be small. For example, in the case of rigid particles, by approximating the integrand in [2.27] by the value for a particle centered at $\mathbf{x}$, setting $\mathbf{y} = \mathbf{x} + (\mathbf{y} - \mathbf{x})$, and expanding in Taylor series similar to [2.6], one finds (Zhang and Prosperetti 1994a; Prosperetti and Zhang 1996):

$$\langle \mathbf{u}_D \rangle(\mathbf{x}, t) = \bar{\mathbf{w}}(\mathbf{x}, t) - \frac{a^2}{10}\left[\nabla^2 \bar{\mathbf{w}} - 2\nabla \times \bar{\boldsymbol{\Omega}} + \frac{1}{n}(\nabla n \cdot \nabla \bar{\mathbf{w}} - 2\nabla n \times \bar{\boldsymbol{\Omega}}) + O\left(\frac{a^2}{L^4}\right)\right], \tag{2.28}$$

where $\boldsymbol{\Omega}$ is the angular velocity vector of the particles around their center of mass. The two terms involving $\bar{\mathbf{w}}$ in the brackets clearly give contributions of order $a^2/L^2$. The estimate of the rotational terms is more problem-dependent (e.g. heavy particles released with a high angular velocity, external torques, etc.) and an estimate of general validity is not possible. On the other hand, one can readily think of many situations of practical interest in which $\bar{\boldsymbol{\Omega}}$ will be of the order of $\nabla \times \langle \mathbf{u}_C \rangle$ and, therefore, of order $1/L$. In all such cases the difference between $\mathbf{w}$ and $\langle \mathbf{u}_D \rangle$ will then be of order $a^2/L^2$. As another example, for uniformly distributed non-interacting spherical drops translating in a quiescent liquid in the Stokes regime, one finds exactly

$$\langle \mathbf{u}_D \rangle(\mathbf{x}, t) = \bar{\mathbf{w}}(\mathbf{x}, t). \tag{2.29}$$

The phase velocity $\langle \mathbf{u}_D \rangle$ satisfies a continuity equation analogous to [2.21], namely

$$\frac{\partial \epsilon_D}{\partial t} + \nabla \cdot (\epsilon_D \langle \mathbf{u}_D \rangle) = 0. \tag{2.30}$$

This reduces to [2.25] if the difference between $\bar{\mathbf{w}}$ and $\langle \mathbf{u}_D \rangle$ is disregarded and $\epsilon_D$ is set equal to $nv$. With the same approximations, the mean volumetric flow rate of the mixture, $\mathbf{u}_m$, is

$$\mathbf{u}_m \equiv \epsilon_C \langle \mathbf{u}_C \rangle + \epsilon_D \langle \mathbf{u}_D \rangle \simeq \epsilon_C \langle \mathbf{u}_C \rangle + \epsilon_D \bar{\mathbf{w}}. \tag{2.31}$$

## 3. AVERAGE STRAIN RATE

The continuous-phase averaged momentum equation can be derived directly by averaging the Navier–Stokes equation and using a relation similar to [2.9] to interchange the Laplace and the averaging operators. In order to study the stress system in the suspension, however, it is expedient to consider first the relation of the average strain rate

$$\langle \mathbf{e}_C \rangle = \mathrm{Symm}\{\langle \nabla \mathbf{u}_C \rangle\} \equiv \tfrac{1}{2}(\langle \nabla \mathbf{u}_C \rangle + \langle \nabla \mathbf{u}_C \rangle^{\mathrm{T}}), \tag{3.1}$$

to the macroscopic flow quantities. Here and in the following Symm{**T**} denotes the symmetric part of the second-order tensor **T**. Let $\mathbf{E}_C$ denote the strain rate of the average velocity field,

$$\mathbf{E}_C(\mathbf{x}) = \text{Symm}\{\nabla\langle\mathbf{u}_C\rangle\}. \tag{3.2}$$

By using [2.13] and [2.16], correct to the order $(a/L)$ required in subsequent developments, we have

$$\langle\mathbf{e}_C\rangle(\mathbf{x}, t) = \mathbf{E}_C - \frac{1}{\epsilon_C}\text{Symm}\{\epsilon_D\mathscr{A}[\mathbf{u}_C] - \nabla\cdot(\epsilon_D\mathscr{T}[\mathbf{u}_C]) - \nabla\nabla : (\epsilon_D\mathscr{S}[\mathbf{u}_C])\}. \tag{3.3}$$

According to this relation (that, being kinematic in nature, holds for general flows), the average strain rate depends not only on the average velocity field, but, as expected, also on the local particle-induced velocity fluctuations.

As is evident from the explicit expressions [2.14] and [2.17], the reduction of [3.3] to an explicit form requires information about the conditionally-averaged field $\langle\mathbf{u}_C\rangle_1$. The case of rigid particles is particularly simple as, due to the no-slip condition at the particle surface, we may write, for the particle centered at **x**,

$$\mathbf{u}_C(\mathbf{z}, t; N) = \mathbf{w} + \boldsymbol{\Omega}\times(\mathbf{z} - \mathbf{x}), \tag{3.4}$$

where $\boldsymbol{\Omega}$ is the angular velocity about the center. Upon taking the one-particle conditional average we simply find

$$\langle\mathbf{u}_C\rangle_1(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w}) = \mathbf{w} + \boldsymbol{\Omega}\times(\mathbf{z} - \mathbf{x}), \tag{3.5}$$

from which, exactly,

$$\text{Symm}\{\mathscr{A}[\mathbf{u}_C]\} = -\mathbf{E}_C, \tag{3.6}$$

$$\text{Symm}\{\nabla\cdot(\epsilon_D\mathscr{T}[\mathbf{u}_C])\} = \text{Symm}\{\nabla[\epsilon_D(\bar{\mathbf{w}} - \langle\mathbf{u}_C\rangle)]\}. \tag{3.7}$$

The term $\mathscr{S}[\mathbf{u}_C]$ gives contributions of higher order in $a/L$. With these results, [3.3] becomes for rigid particles

$$\langle\mathbf{e}_C\rangle = \frac{1}{\epsilon_C}(\mathbf{E}_C + \text{Symm}\{\nabla[\epsilon_D(\bar{\mathbf{w}} - \langle\mathbf{u}_C\rangle)]\}). \tag{3.8}$$

An alternative equivalent form is

$$\langle\mathbf{e}_C\rangle = \frac{1}{\epsilon_C}\mathbf{E}_m, \tag{3.9}$$

where $\mathbf{E}_m$ is the strain rate of the mean velocity field $\mathbf{u}_m$ defined by [2.31]:

$$\mathbf{E}_m = \text{Symm}\{\nabla\mathbf{u}_m\}. \tag{3.10}$$

The result [3.8] is as expected because, as is readily verified, due to the continuity of velocities across the interface,

$$\mathbf{E}_m = \epsilon_C\langle\mathbf{e}_C\rangle + \epsilon_D\langle\mathbf{e}_D\rangle, \tag{3.11}$$

and the disperse-phase microscopic strain rate $\mathbf{e}_D$ vanishes identically for rigid particles.

## 4. AVERAGED MOMENTUM EQUATIONS

We now proceed to derive the momentum equations in the 'two-fluid' form widely used in engineering. We use the phase average [2.8] for the continuous and the 'particle' average [2.22] for the disperse phase. A derivation of the momentum equation of the latter in terms of phase averages is given in appendix A.

### *4.1. Continuous phase*

We write the momentum equation for the continuous phase as

$$\frac{\partial\rho_C\mathbf{u}_C}{\partial t} + \nabla\cdot(\rho_C\mathbf{u}_C\mathbf{u}_C) = \nabla\cdot\boldsymbol{\sigma}_C + \rho_C\mathbf{g}, \tag{4.1}$$

where $\rho_C$ is the density, $\mathbf{g}$ is the body force, and $\boldsymbol{\sigma}_C$ is the stress tensor. Upon using [2.13] and [2.20] the averaged form of this equation is

$$\frac{\partial}{\partial t}(\epsilon_C \rho_C \langle \mathbf{u}_C \rangle) + \nabla \cdot (\rho_C \epsilon_C \langle \mathbf{u}_C \mathbf{u}_C \rangle) = \epsilon_C \nabla \cdot \langle \boldsymbol{\sigma}_C \rangle - \epsilon_D \mathscr{A}[\boldsymbol{\sigma}_C] + \nabla \cdot (\epsilon_D \mathscr{L}[\boldsymbol{\sigma}_C]) + \rho_C \epsilon_C \mathbf{g}. \quad [4.2]$$

Upon writing explicitly the term $\mathscr{A}[\boldsymbol{\sigma}_C]$ according to the definition [2.14] we have

$$\mathscr{A}[\boldsymbol{\sigma}_C] = \frac{1}{\epsilon_D} \int d^3w P(\mathbf{x}, \mathbf{w}; t) \int_{|\mathbf{z}-\mathbf{x}|=a} dS_{\mathbf{z}}(\langle \boldsymbol{\sigma}_C \rangle_1 - \langle \boldsymbol{\sigma}_C \rangle) \cdot \mathbf{n}. \quad [4.3]$$

With the approximation [2.6], $\epsilon_D \simeq nv$, the first part of this term will be recognized as the average continuous-phase force $\bar{\mathbf{f}}/v$ per unit particle volume acting on the particle centered at $\mathbf{x}$. The second part can be simplified by a Taylor series expansion around $\mathbf{x}$ to find, with an error of order $a^2/L^2$,

$$\mathscr{A}[\boldsymbol{\sigma}_C] = \frac{1}{v} \bar{\mathbf{f}} - \nabla \cdot \langle \boldsymbol{\sigma}_C \rangle. \quad [4.4]$$

In our earlier papers (Zhang and Prosperetti 1994a, 1994b) it was sufficient to truncate the expansion [2.16] of the tensor $\mathscr{L}[\boldsymbol{\sigma}_C]$ to its first term, namely

$$\mathscr{T}[\boldsymbol{\sigma}_C] = \frac{a}{\epsilon_D} \int P(\mathbf{x}, \mathbf{w}; t)\, d^3w \int_{|\mathbf{z}-\mathbf{x}|=a} dS_{\mathbf{z}}[(\langle \boldsymbol{\sigma}_C \rangle_1 - \langle \boldsymbol{\sigma}_C \rangle) \cdot \mathbf{n}]\mathbf{n}. \quad [4.5]$$

Proceeding as before, it is easy to show that, with an error of order $(a^2/L)^2$,

$$\mathscr{T}[\boldsymbol{\sigma}_C] = \frac{1}{v} \bar{\mathbf{s}} - \langle \boldsymbol{\sigma}_C \rangle, \quad [4.6]$$

where $\bar{\mathbf{s}}$ is the average stresslet-torque per particle, i.e. the first moment of the surface traction about the particle center. The physical concreteness of this term is highlighted, e.g. by the fact that, in a suspension of bubbles, the trace of $\mathscr{T}[\boldsymbol{\sigma}_C]$ appears as a part of the 'ambient' pressure in the modified Rayleigh–Plesset equation valid in this case (Zhang and Prosperetti 1994b). The other terms in the expansion [2.16] of $\mathscr{L}[\boldsymbol{\sigma}_C]$ are related to higher moments about the particle center of the surface traction.

Upon introducing the 'kinematic' Reynolds stress $\mathbf{M}_C$ by the definition

$$\mathbf{M}_C = \langle \mathbf{u}_C \rangle \langle \mathbf{u}_C \rangle - \langle \mathbf{u}_C \mathbf{u}_C \rangle = -\langle (\mathbf{u}_C - \langle \mathbf{u}_C \rangle)(\mathbf{u}_C - \langle \mathbf{u}_C \rangle) \rangle, \quad [4.7]$$

and using the continuity equation [2.21], we can rewrite the momentum equation [4.2] in the form

$$\rho_C \epsilon_C \left( \frac{\partial \langle \mathbf{u}_C \rangle}{\partial t} + \langle \mathbf{u}_C \rangle \cdot \nabla \langle \mathbf{u}_C \rangle \right) = \epsilon_C \nabla \cdot \langle \boldsymbol{\sigma}_C \rangle - \epsilon_D \mathscr{A}[\boldsymbol{\sigma}_C] + \nabla \cdot (\epsilon_C \rho_C \mathbf{M}_C + \epsilon_D \mathscr{L}[\boldsymbol{\sigma}_C]) + \rho_C \epsilon_C \mathbf{g}. \quad [4.8]$$

The first four terms have the structure that is expected intuitively: the volume occupied by the continuous-phase per unit mixture volume is $\epsilon_C$ and the particles act as a source of momentum of the order of their number density times the force per particle. The Reynolds transport term is similarly expected. The term $\mathscr{L}$, showing that the particles act not only as a source of momentum, but also contribute to its transport, is perhaps less obvious. Its physical origin is rooted in the finite size of the particles.

For an incompressible, Newtonian continuous phase, $\boldsymbol{\sigma}_C = -p_C \mathbf{I} + 2\mu_C \mathbf{e}_C$, where $p_C$ is the pressure, $\mu_C$ the viscosity coefficient, and $\mathbf{I}$ the identity tensor, so that $\langle \boldsymbol{\sigma}_C \rangle = -\langle p_C \rangle \mathbf{I} + 2\mu_C \langle \mathbf{e}_C \rangle$ with $\langle \mathbf{e}_C \rangle$ given by [3.3].

*4.2. Disperse phase*

The equation of motion for the particles, all with equal mass $m$, is

$$m\dot{\mathbf{w}}(\mathbf{x}, t) = \int_{|\mathbf{x}-\mathbf{z}|=a} \mathrm{d}S_z \boldsymbol{\sigma}_C(\mathbf{z}, t; N) \cdot \mathbf{n} + \mathbf{f}_c + m\mathbf{g}, \qquad [4.9]$$

where $\mathbf{f}_c$ is the collision force. Upon averaging this equation according to [2.22] and use of the transport theorem [2.24], one finds

$$m\frac{\partial n\bar{\mathbf{w}}}{\partial t} + m\nabla \cdot (n\overline{\mathbf{ww}}) = \int \mathrm{d}^3 w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{x}-\mathbf{z}|=a} \mathrm{d}S_z \langle \boldsymbol{\sigma}_C \rangle_1 \cdot \mathbf{n} + \nabla \cdot \boldsymbol{\sigma}_c + nm\mathbf{g}. \qquad [4.10]$$

The term $\boldsymbol{\sigma}_c$ is the collision stress due to direct particle–particle interaction derived in appendix C (see also Sangani and Didwania 1993a; Zhang and Rauenzahn 1997). It should be noted that the derivation given in appendix C does not presuppose binary or short-duration collisions.

This relation expresses the momentum balance for all the particles whose center is inside the unit volume. An alternative momentum equation giving the momentum balance for all the particle material entirely inside the unit volume can be obtained by performing the phase average of [2.8]. This derivation is given in appendix A. Both relations are of course correct and their difference is due to the fact that they refer to different systems. However it may be expected that, whenever a description in terms of averaged quantities is justified, the differences between the two descriptions will be small. We return on this point in appendix A.

To put [4.10] in a more useful form, we note that

$$\int \mathrm{d}^3 w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{x}-\mathbf{z}|=a} \mathrm{d}S_z \langle \boldsymbol{\sigma}_C \rangle_1 \cdot \mathbf{n} = \epsilon_D \mathscr{A}[\boldsymbol{\sigma}_C] + \int \mathrm{d}^3 w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{x}-\mathbf{z}|=a} \mathrm{d}S_z \langle \boldsymbol{\sigma}_C \rangle(\mathbf{z}, t) \cdot \mathbf{n}$$

$$= \epsilon_D \mathscr{A}[\boldsymbol{\sigma}_C] + nv\nabla \cdot \langle \boldsymbol{\sigma}_C \rangle(\mathbf{x}, t) + O\left(\epsilon_D \left(\frac{a}{L}\right)^2 \langle \boldsymbol{\sigma}_C \rangle\right), \qquad [4.11]$$

where $\mathscr{A}[\boldsymbol{\sigma}_C]$ is defined in [4.3] and $\langle \boldsymbol{\sigma}_C \rangle(\mathbf{z}, t)$ has been expanded in Taylor series around $\mathbf{x}$ thereby introducing an error of order $(a^2/L^2)$. Dropping the error of the same order arising by approximating $\epsilon_D$ by $nv$, we may then rewrite the momentum equation [4.10] as

$$\rho_D \epsilon_D \left(\frac{\partial \bar{\mathbf{w}}}{\partial t} + \bar{\mathbf{w}} \cdot \nabla \bar{\mathbf{w}}\right) = \epsilon_D \nabla \cdot \langle \boldsymbol{\sigma}_C \rangle + \epsilon_D \mathscr{A}[\boldsymbol{\sigma}_C] + \rho_D \nabla \cdot (\epsilon_D \mathbf{M}_D) + \nabla \cdot \boldsymbol{\sigma}_c + \rho_D \epsilon_D \mathbf{g}, \qquad [4.12]$$

where $\rho_D$ is the density of the particle material. Here we have used the number density conservation equation [2.25] and we have introduced the kinematic Reynolds stress for the disperse phase $\mathbf{M}_D$ defined by

$$\mathbf{M}_D = \overline{\mathbf{ww}} - \overline{\mathbf{w}}\,\overline{\mathbf{w}} = -\overline{(\mathbf{w} - \bar{\mathbf{w}})(\mathbf{w} - \bar{\mathbf{w}})}. \qquad [4.13]$$

It will be noted that [4.12] explicitly shows the particles to respond to the *continuous-phase* pressure, rather than to some disperse-phase pressure. This feature is according to physical intuition. The disperse-phase pressure, i.e. the pressure inside the particles, cannot affect the motion of the particles directly, but only indirectly through its relation with the continuous-phase pressure resulting from dynamical boundary conditions at the particles surface (see also appendix A and Prosperetti and Zhang 1996). This rather involved conceptual circle is avoided by the above procedure. Furthermore, there are cases where a disperse-phase pressure cannot even be meaningfully defined, as for rigid particles (see, e.g. Givler 1993). Of course, [4.12] does not imply that further information on the internal dynamics of the particles is not necessary, as this will be needed to calculate the conditionally-averaged fields. However, the type of information required can be tailored to the problem. For example, the case of rigid particles can be addressed directly rather than as the limit of stiffer and stiffer particles (Drew and Lahey 1994). Similarly, in the case

of gas bubbles, there is no need to solve the momentum equation in the gas, but only to state, for example, that the gas pressure is spatially uniform inside each bubble.

*4.3. Disperse-phase angular momentum*

The averaged particle angular momentum equation can be derived by a similar procedure. If the inertia of the rotational motion is considered, the phase space must be extended to include the angular velocities $\boldsymbol{\Omega}$ of the particles. Correspondingly, the integrals $\int \mathrm{d}^3w$ should be changed to $\int \mathrm{d}^3w \int \mathrm{d}^3\Omega$.

The angular momentum equation for each particle is

$$J\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} = \int_{|\mathbf{x}-\mathbf{x}|=a} \mathrm{d}S_{\mathbf{z}}(\mathbf{z}-\mathbf{x}) \times (\boldsymbol{\sigma}_{\mathrm{C}} \cdot \mathbf{n}) + \mathbf{L} + \mathbf{m}_{\mathrm{c}}, \tag{4.14}$$

where $J$ is the moment of inertia of the particle, $\mathbf{L}$ the external torque acting on it, and $\mathbf{m}_{\mathrm{c}}$ the torque due to collisions. Upon averaging this equation according to [2.23] and following the same procedure used to derive [4.10], we find

$$J\left[\frac{\partial n\overline{\boldsymbol{\Omega}}}{\partial t} + \nabla(n\overline{\boldsymbol{\Omega}\mathbf{w}})\right] = a\int \mathrm{d}^3w \int \mathrm{d}^3\Omega P(\mathbf{x}, \mathbf{w}, \boldsymbol{\Omega}; t) \int_{|\mathbf{x}-\mathbf{z}|=a} \mathrm{d}S_{\mathbf{z}}\mathbf{n} \times (\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1 \cdot \mathbf{n}) + n\bar{\mathbf{L}} + \bar{\mathbf{m}}_{\mathrm{c}}. \tag{4.15}$$

The integral in the right-hand side can be connected to the skew-symmetric part of $\mathscr{T}[\boldsymbol{\sigma}_{\mathrm{C}}]$ similarly to the calculation done above in [4.11]. The result is

$$a\epsilon_{ijk}\int \mathrm{d}^3w \int \mathrm{d}^3\Omega P(\mathbf{x}, \mathbf{w}, \boldsymbol{\Omega}; t) \int_{|\mathbf{x}-\mathbf{z}|=a} \mathrm{d}S_{\mathbf{z}}(\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1 \cdot \mathbf{n})_j n_k = nv\epsilon_{ijk}(\mathscr{T}[\boldsymbol{\sigma}_{\mathrm{C}}])_{jk} + O\left(\frac{a}{L}\right)^2. \tag{4.16}$$

With the neglect of the error term and use of the continuity equation [2.25], the average angular momentum balance for the disperse phase becomes

$$nJ\left[\frac{\partial\overline{\boldsymbol{\Omega}}_i}{\partial t} + \bar{\mathbf{w}} \cdot \nabla\overline{\boldsymbol{\Omega}}_i\right] = -\epsilon_{\mathrm{D}}\epsilon_{ijk}(\mathscr{T}[\boldsymbol{\sigma}_{\mathrm{C}}])_{jk} + n\bar{L}_i + nJ\frac{\partial M_{\Omega ij}}{\partial x_j} + \bar{\mathbf{m}}_{\mathrm{c}}, \tag{4.17}$$

in which

$$\mathbf{M}_{\Omega} = \overline{\boldsymbol{\Omega}}\,\overline{\mathbf{w}} - \overline{\boldsymbol{\Omega}\mathbf{w}}. \tag{4.18}$$

Since torques do not satisfy an action–reaction principle, unlike the linear momentum equation [4.10], the mean torque due to collisions cannot be written as a divergence.

The momentum equations [4.8] and [4.12] and the continuity equations [2.21] and [2.25] constitute a general two-fluid model for disperse two-phase flows. To close the system one needs constitutive relations expressing the integrals $\mathscr{A}$, etc. in terms of average quantities. In principle these relations should be obtained by solving appropriate equations for the one-particle conditionally averaged fields which however, as is well known, involve other higher-order conditionally averaged fields and are plagued by divergent integrals (see Hinch 1977; Sangani 1991). The only simple case is the dilute limit, with results correct to first order in the particle volume fraction $\epsilon_{\mathrm{D}}$, that we address in the next section for low particle Reynolds number. The opposite limit of potential flow has been studied in Zhang and Prosperetti (1994a,b). It is shown in Zhang (1997) that, for the viscous case, the process can be continued to the next order in $\epsilon_{\mathrm{D}}$ reducing the problem to that studied by Hinch (1977). It appears doubtful that much progress can be made analytically beyond these results. A more promising approach seems to be the direct calculation of the integrals $\mathscr{A}$, etc. by direct numerical simulation. For example, the mean force $\bar{\mathbf{f}}$ and stresslet-torque $\bar{\mathbf{s}}$ appearing in [4.4], [4.6] are quantities routinely calculated in Stokesian dynamics (see, e.g. Brady and Bossis 1988). An example of the application of direct numerical simulation to the linear potential flow problem is given in Zhang and Prosperetti (1994a).

## 5. CLOSURE FOR DILUTE SUSPENSIONS

In order to obtain results correct to first order in the disperse phase volume fraction $\epsilon_D$, one needs the conditionally averaged fields correct to $O(1)$. It is readily shown that, similarly to [2.9], the average of the gradient of any one-particle conditionally-averaged field differs from the gradient of the average by a term $O(\epsilon_D)$. For an accuracy of order 1 this difference can therefore be disregarded, which leads to equations for $\langle \mathbf{u}_C \rangle_1$ and $\langle p_C \rangle_1$ identical in form to those satisfied by the exact, unaveraged fields, i.e. [4.1]. With this truncation, the one-particle conditionally averaged equations form a closed system and can be solved.

Here we carry out this procedure under the assumption that the Reynolds number for the *relative* motion of the individual particles and the fluid is small. This will be a good approximation for small particles and it does not imply that the global flow on the macroscopic scale $L$ satisfies the Stokes equations.

The problem to be solved for the one-particle conditionally averaged fields is

$$-\nabla \langle p_C \rangle_1 + \mu_C \nabla^2 \langle \mathbf{u}_C \rangle_1 = 0, \tag{5.1}$$

$$\nabla \cdot \langle \mathbf{u}_C \rangle_1 = \mathbf{0}. \tag{5.2}$$

The solution of these equations is subject to the condition [3.5] at the particle surface while, at large distance from the particle, it must satisfy

$$\langle \mathbf{u}_C \rangle_1(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w}) \to \langle \mathbf{u}_C \rangle(\mathbf{x}, t) + (\mathbf{z} - \mathbf{x}) \cdot \nabla \langle \mathbf{u}_C \rangle(\mathbf{x}, t) + \tfrac{1}{2}(\mathbf{z} - \mathbf{x})(\mathbf{z} - \mathbf{x}) : \nabla\nabla \langle \mathbf{u}_C \rangle(\mathbf{x}, t). \tag{5.3}$$

The last two terms in the right hand side are necessary to obtain a result accurate to $O(a/L)$. This relation is similar to that used by Hinch (1977). Its justification ultimately rests on the assumed separation of scales, i.e. $a \ll L$. The influence of the fixed particle must decrease as $|\mathbf{z} - \mathbf{x}|$ becomes large on the scale $a$, but over such distances the unconditional average $\langle \mathbf{u}_C \rangle$ (that varies over the scale $L$), is still close to its value at the particle center $\mathbf{x}$.

The problem for the conditionally averaged fields is therefore formally the same as that for the flow around an isolated spherical particle in the ambient flow [5.3]. The particles interact only through the average fields and direct particle–particle effects are negligible.

For rigid particles, the average of the microscopic strain has been given before in [3.8] or [3.9]. By using [4.4] and the Faxén theorem result for the force on a rigid particle in an arbitrary ambient flow (see, e.g. Kim and Karrila 1991, p. 78) we find, accurate to $O(1)$ in $\epsilon_D$,

$$\mathscr{A}[\boldsymbol{\sigma}_C] = -\frac{9}{2a^2} \mu_C (\bar{\mathbf{w}} - \mathbf{u}_C)) + \tfrac{3}{4} \mu_C \nabla^2 \langle \mathbf{u}_C \rangle. \tag{5.4}$$

Similarly, for $\mathscr{T}[\boldsymbol{\sigma}_C]$, we use [4.5] and find

$$(\mathscr{T}[\boldsymbol{\sigma}_C])_{jk} = 3\mu_C (\mathbf{E}_C)_{jk} + 3\mu_C \epsilon_{jkl} (\bar{\boldsymbol{\Omega}} - \tfrac{1}{2} \nabla \times \langle \mathbf{u}_C \rangle)_l. \tag{5.5}$$

The last two terms in [5.3] give contributions of higher order in $a/L$ to $\mathscr{S}[\boldsymbol{\sigma}_C]$ and $\mathscr{R}[\boldsymbol{\sigma}_C]$ which are themselves higher-order corrections and can, therefore, be calculated by using the well-known solution for a particle immersed in a uniform flow (see, e.g. Batchelor 1967; Kim and Karrila 1991). The result is

$$\epsilon_D (\mathscr{S}[\boldsymbol{\sigma}_C])_{ijk} = \tfrac{3}{4} \mu_C \epsilon_D (\bar{w}_i - \langle u_{Ci} \rangle) \delta_{jk}. \tag{5.6}$$

Due to the particular symmetry of $\langle \boldsymbol{\sigma}_C \rangle_1$, for spherical particles in the dilute limit, the contribution of $\mathscr{R}[\boldsymbol{\sigma}_C]$ is found to be $O(a^2/L^2)$ and can, therefore, be neglected. For particles of different shape or at higher concentrations, however, this term gives a contribution of order $a/L$ and must be retained for consistency.

The final quantity to be evaluated to close the momentum equations is the Reynolds stress $\mathbf{M}_C$. This is a complex matter on which we have little to offer beyond the following comments. One can envisage situations (e.g. particles in a very viscous liquid) in which the only element of randomness at any given time is the position of the particles. In this case significant differences between the local instantaneous velocity $\mathbf{u}_C(\mathbf{x}, t; N)$ and its ensemble average value $\langle \mathbf{u}_C \rangle(\mathbf{x}, t)$ can only be due to the presence of a nearby particle, and these differences must be of the order of the

particle–fluid relative velocity. This is a small quantity due to the assumption of small particle Reynolds number and, since $\mathbf{M}_C$ is quadratic in this difference, it can be neglected consistently with the other approximations introduced. It is also possible, however, that different particle arrangements cause large-scale convective motions in the mixture or that the flow is turbulent. Clearly, in these situations, $\mathbf{M}_C$ cannot be neglected. Similar arguments apply to $\mathbf{M}_D$.

Upon substituting the previous results into the general forms [4.8], [4.12] of the momentum equations we then have

$$\epsilon_C\rho_C\left(\frac{\partial\langle\mathbf{u}_C\rangle}{\partial t}+\langle\mathbf{u}_C\rangle\cdot\nabla\langle\mathbf{u}_C\rangle\right)=-\epsilon_C\nabla\langle p_C\rangle+\epsilon_C\rho_C\mathbf{g}+\epsilon_C\boldsymbol{\nabla}\cdot(2\mu^*\mathbf{E}_m)+\frac{9\epsilon_D\mu_C}{2a^2}(\bar{\mathbf{w}}-\langle\mathbf{u}_C\rangle)$$

$$-\tfrac{3}{4}\epsilon_D\mu_C\nabla^2\langle\mathbf{u}_C\rangle+\tfrac{3}{4}\mu_C\nabla^2[\epsilon_D(\bar{\mathbf{w}}-\langle\mathbf{u}_C\rangle)]+\boldsymbol{\nabla}\cdot(\epsilon_C\mathbf{M}_C)+3\mu_C\nabla\times[\epsilon_D(\bar{\boldsymbol{\Omega}}-\tfrac{1}{2}\nabla\times\langle\mathbf{u}_C\rangle)] \quad [5.7]$$

$$\epsilon_D\rho_D\left(\frac{\partial\bar{\mathbf{w}}}{\partial t}+\bar{\mathbf{w}}\cdot\nabla\bar{\mathbf{w}}\right)=-\epsilon_D\nabla\langle p_C\rangle+2\mu_C\epsilon_D\boldsymbol{\nabla}\cdot\mathbf{E}_C$$

$$-\frac{9\mu_C\epsilon_D}{2a^2}(\bar{\mathbf{w}}-\langle\mathbf{u}_C\rangle)+\tfrac{3}{4}\mu_C\epsilon_D\nabla^2\langle\mathbf{u}_C\rangle+\boldsymbol{\nabla}\cdot(\epsilon_D\mathbf{M}_D)+\epsilon_D\rho_D\mathbf{g}, \quad [5.8]$$

where $\mathbf{E}_C$ is given by [3.2] and $\mu^*$ is the well-known effective viscosity of a dilute suspension of particles (see, e.g. Batchelor 1967)

$$\frac{\mu^*}{\mu_C}=1+\tfrac{5}{2}\epsilon_D+o(\epsilon_D). \quad [5.9]$$

The two momentum equations bear of course a strong similarity to those derived by others, but also exhibit some important differences. In the first place, since our method of derivation is applicable to the spatially non-uniform case, the relative position of $\epsilon_D$ and the spatial differentiation operators should be noticed. Secondly, we do find the standard Einstein viscosity correction, but we also have the term $\frac{3}{4}\mu_C\nabla^2[\epsilon_D(\bar{\mathbf{w}}-\langle\mathbf{u}_C\rangle)]$ that appears to be new. In principle, this term confers a non-Newtonian nature to the stress-strain relation although it must be recognized that, in many practical circumstances, the difference $(\bar{\mathbf{w}}-\langle\mathbf{u}_C\rangle)$ may be so small that this term plays a minor role.

The two momentum equations can be combined to form a mixture momentum equation that will be discussed in section 8. The corresponding results for a suspension of viscous drops are derived in appendix B.

For fixed particles, which would be the case in a porous medium, with the neglect of the Reynolds stress and of the inertia of the continuous phase, the momentum equation [5.7] becomes

$$\nabla\langle p_C\rangle=\mu_C(1+\tfrac{3}{2}\epsilon_D)\nabla^2\langle\mathbf{u}_C\rangle-\frac{9\epsilon_D\mu_C}{2a^2}\langle\mathbf{u}_C\rangle-\tfrac{7}{4}\mu_C\langle\mathbf{u}_C\rangle\nabla^2\epsilon_D$$

$$+\mu_C(\nabla\langle\mathbf{u}_C\rangle)^{\mathrm{T}}\cdot\nabla\epsilon_D+\tfrac{1}{2}\mu_C(\nabla\langle\mathbf{u}_C\rangle)\cdot\nabla\epsilon_D+\rho_C\mathbf{g}, \quad [5.10]$$

where $[(\nabla\langle\mathbf{u}_C\rangle)^{\mathrm{T}}\cdot\nabla\epsilon_D]_i=(\partial\epsilon_D/\partial x_j)(\partial\langle u_{Cj}\rangle/\partial x_i)$. In particular, for a uniform particle distribution, in terms of the volumetric flow rate $\mathbf{u}_m=\epsilon_C\langle\mathbf{u}_C\rangle$,

$$\nabla\langle p_C\rangle=(1+\tfrac{5}{2}\epsilon_D)\mu_C\nabla^2\mathbf{u}_m-\frac{9\epsilon_D\mu_C}{2a^2}\mathbf{u}_m+\rho_C\mathbf{g} \quad [5.11]$$

in agreement with Brinkman's form (1947). The diffusive term is seen to require Einstein's expression [5.9] for the effective viscosity. Recent numerical work (Martys *et al.* 1994) seems to substantiate earlier experimental indications that this may not be so at finite volume fractions. Equations [5.10] represents a non-trivial extension of Brinkman's equation to the case of a non-uniform medium.

To close the system one must add to the two linear momentum equations [5.7], [5.8] the equation for the angular momentum of the particles. Neglecting the rotational Reynolds stress [4.18] and

collisional effects, that are negligible to first order in $\epsilon_D$, the general angular momentum equation becomes, with [5.5],

$$nJ\left[\frac{\partial\bar{\boldsymbol{\Omega}}}{\partial t}+\bar{\mathbf{w}}\cdot\nabla\bar{\boldsymbol{\Omega}}\right]=6\epsilon_D\mu_C(\tfrac{1}{2}\nabla\times\langle\mathbf{u}_C\rangle-\bar{\boldsymbol{\Omega}})+n\bar{\mathbf{L}}. \tag{5.12}$$

In evaluating each one of the quantities substituted into these equations (e.g. $\mathscr{A}[\boldsymbol{\sigma}_C]$, $\mathscr{T}[\boldsymbol{\sigma}_C]$, etc.), only terms of order $\epsilon_D$ and $a/L$ have been retained. Strictly speaking, however, the final equations may be inconsistent in a problem-dependent way in a sense that is best explained with an example. Suppose that, in a particular flow, the terms dropped in the expression [5.4] for $\mathscr{A}[\boldsymbol{\sigma}_C]$ happen to be of the same order of magnitude as those retained for $S[\boldsymbol{\sigma}_C]$ in [5.6]. Then, in a formally consistent asymptotic analysis, also $\mathscr{S}[\boldsymbol{\sigma}_C]$ should be dropped. While this is a valid point, the view that we take is that we retain the leading-order contributions for all the terms. Should in some cases some of these contributions be as small as those neglected, the error incurred will also be small. Clearly, this is the only possible approach if the intent is—as in the present paper—to derive equations of general applicability to a variety of flows.

## 6. AVERAGED ENERGY EQUATIONS

The averaged energy equations for the system considered in this paper can be derived in the same way as the momentum equations in section 4. In this case the probability space may need to be enlarged to include additional degrees of freedom related to the temperature field in the particles.

For the continuous phase the microscopic energy equation is

$$\rho_C C_{pC}\left[\frac{\partial T_C}{\partial t}+\nabla\cdot(T_C\mathbf{u}_C)\right]=-\nabla\cdot\mathbf{q}_C, \tag{6.1}$$

where $C_{pC}$ is the specific heat and $\mathbf{q}_C$ the heat flux. For an isotropic material

$$\mathbf{q}_C=-K_C\nabla T_C, \tag{6.2}$$

where $K_C$ is the thermal conductivity. By using the transport theorem [2.20] together with [2.13], the averaged energy equation can be written as

$$\rho_C C_{pC}\left[\frac{\partial(\epsilon_C\langle T_C\rangle)}{\partial t}+\nabla\cdot(\epsilon_C\langle T_C\mathbf{u}_C\rangle)\right]=-\epsilon_C\nabla\cdot\langle\mathbf{q}_C\rangle+\epsilon_D\mathscr{A}[\mathbf{q}_C]-\nabla\cdot(\epsilon_D\mathscr{L}[\mathbf{q}_C]), \tag{6.3}$$

where $\mathscr{A}[\mathbf{q}_C]$ and $\mathscr{L}[\mathbf{q}_C]$ are given by [2.14], [2.15]. The quantity $\mathscr{A}[\mathbf{q}_C]$ is an effective heat source while $\mathscr{L}[\mathbf{q}_C]$ has the nature of an additional contribution to the heat flux. As before, the evaluation of these terms requires the one-particle conditionally averaged fields.

If we introduce the fluctuating contribution to the convective heat flux $\mathbf{Q}_C$ by

$$\mathbf{Q}_C=\rho_C C_{pC}(\langle T_C\rangle\langle\mathbf{u}_C\rangle-\langle T_C\mathbf{u}_C\rangle)=-\rho_C C_{pC}\langle(T_C-\langle T_C\rangle)(\mathbf{u}_C-\langle\mathbf{u}_C\rangle)\rangle, \tag{6.4}$$

and use the continuity equation [2.21], the averaged energy equation [6.3] can be written as

$$\rho_C C_{pC}\epsilon_C\left(\frac{\partial\langle T_C\rangle}{\partial t}+\langle\mathbf{u}_C\rangle\cdot\nabla\langle T_C\rangle\right)=-\epsilon_C\nabla\cdot\langle\mathbf{q}_C\rangle+\nabla\cdot(\epsilon_C\mathbf{Q}_C-\epsilon_D\mathscr{L}[\mathbf{q}_C])+\epsilon_D\mathscr{A}[\mathbf{q}_C]. \tag{6.5}$$

Equation [2.15] can also be used to express the average heat flux in the form

$$\langle\mathbf{q}_C\rangle=-\langle K_C\nabla T_C\rangle=-K_C\nabla\langle T_C\rangle+\frac{K_C}{\epsilon_C}\{\epsilon_D\mathscr{A}[T_C]-\nabla\cdot(\epsilon_D\mathscr{L}[T_C])\}, \tag{6.6}$$

or, from [2.16],

$$\langle\mathbf{q}_C\rangle=-K_C\nabla\langle T_C\rangle+\frac{K_C}{\epsilon_C}\{\epsilon_D\mathscr{A}[T_C]-\nabla\cdot[\epsilon_D\mathscr{T}[T_C]+\nabla\cdot(\epsilon_D\mathscr{S}[T_C])]\}+O\left(\frac{a^2}{L^2}\epsilon_D\langle T_C\rangle_1\right). \tag{6.7}$$

We now turn to the disperse phase. Let $e_D$ be the internal energy of a particle. Then we have

$$\frac{de_D}{dt} = -\int_{|z-x|=a} dS_z \mathbf{q}_C \cdot \mathbf{n} + vs_D, \tag{6.8}$$

where $s_D$ is the volume-averaged heat source density in the particle. The particle volume-averaged temperature is defined by

$$T_D = \frac{e_D}{\rho_D C_{pD} v}. \tag{6.9}$$

Upon taking $g = e_D$ in the transport equation [2.24], we have the averaged equation for the disperse phase as

$$\left[\frac{\partial n\bar{e}_D}{\partial t} + \nabla \cdot (n\overline{\mathbf{w}e_D})\right] = -\int d^3w P(\mathbf{x}, \mathbf{w}, t) \int_{|z-x|=a} \langle \mathbf{q}_C \rangle_1 \cdot \mathbf{n}\, dS_z + nv\bar{s}_D. \tag{6.10}$$

By proceeding as in [4.11], this equation can be written as

$$\left[\frac{\partial n\bar{e}_D}{\partial t} + \nabla \cdot (n\overline{\mathbf{w}e_D})\right] = -\epsilon_D \mathscr{A}[\mathbf{q}_C] - \epsilon_D \nabla \cdot \langle \mathbf{q}_C \rangle + \epsilon_D \bar{s}_D, \tag{6.11}$$

with an error of order $a^2/L^2$.

By introducing the fluctuating heat flux similar to [6.4]

$$\mathbf{Q}_D = (\bar{e}_D\tilde{\mathbf{w}} - \overline{e_D\mathbf{w}}) = -\overline{(e_D - \bar{e}_D)(\mathbf{w} - \tilde{\mathbf{w}})}, \tag{6.12}$$

and using the continuity equation [2.25], the average energy equation for the disperse phase becomes

$$n\left(\frac{\partial \bar{e}_D}{\partial t} + \tilde{\mathbf{w}} \cdot \nabla \bar{e}_D\right) = -\epsilon_D \mathscr{A}[\mathbf{q}_C] - \epsilon_D \nabla \cdot \langle \mathbf{q}_C \rangle + \nabla \cdot (n\mathbf{Q}_D) + \epsilon_D \bar{s}_D. \tag{6.13}$$

This equation can also be expressed in terms of $\overline{T}_D$, the particle ensemble-average of $T_D$. A formulation in terms of the phase average temperature $\langle T_D \rangle$ is given in appendix A.

The previous results can readily be specialized to the case of pure conduction by dropping the convective terms in the left-hand side.

## 7. EXAMPLES OF FIRST-ORDER CLOSURE FOR THE ENERGY EQUATION

We now present two simple examples of first-order closure for the energy equation in the case of solid particles. For the same reasons mentioned at the beginning of section 5, to $O(\epsilon_D)$, $\langle T_C \rangle_1$ satisfies the usual convection–diffusion equation

$$\frac{\partial \langle T_C \rangle_1}{\partial t} + \langle \mathbf{u}_C \rangle_1 \cdot \nabla \langle T_C \rangle_1 = D_C \nabla^2 \langle T_C \rangle_1, \tag{7.1}$$

where

$$D_C = \frac{K_C}{\rho_C C_{pC}}, \tag{7.2}$$

is the thermal diffusivity, and the velocity field $\langle \mathbf{u}_C \rangle_1$ is given by [5.1] and [5.2]. In order to find a solution correct to $O(a/L)$, [7.1] must be solved subject to a condition analogous to [5.3], namely

$$\langle T_C \rangle_1(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w}) \to \langle T_C \rangle(\mathbf{x}, t) + (\mathbf{z} - \mathbf{x}) \cdot \nabla \langle T_C \rangle(\mathbf{x}, t) + \tfrac{1}{2}(\mathbf{z} - \mathbf{x})(\mathbf{z} - \mathbf{x}) : \nabla\nabla \langle T_C \rangle(\mathbf{x}, t). \tag{7.3}$$

In the calculations that follow we also need an expression for $\langle \mathbf{q}_C \rangle_1$. This can be derived similarly to [6.6], but to the order of accuracy required here it is simply $\langle \mathbf{q}_C \rangle_1 = -K_C \nabla \langle T_C \rangle_1$.

In both the examples that follow we assume that the time scale $\tau$ of variation of $\langle T_C \rangle$ is such that the thermal penetration length $\sqrt{D_{C,D}\tau}$ in either phase is much greater than the particle radius, although not necessarily much greater than the macroscopic length $L$. With this assumption the local problem is quasi-steady and the time derivative in [7.1] can be disregarded.

*7.1. Steady conduction*

Our first example is pure conduction in the absence of flow so that $\mathbf{u}_C = \mathbf{w} = \mathbf{0}$.

Let $T_D$ be the volume-average temperature of particle 1. We assume that $T_D$ is the result of a spatially uniform volumetric heat source inside the particle and of the matching with the continuous-phase temperature outside. With these assumptions we find

$$\langle T_C \rangle_1(\mathbf{x}+\mathbf{r}, t \mid \mathbf{x}, \mathbf{w}) = \langle T_C \rangle(\mathbf{x}, t) + \left(1 - \frac{K_D - K_C}{K_D + 2K_C}\frac{a^3}{r^3}\right)\mathbf{r}\cdot\nabla\langle T_C\rangle$$

$$+ \frac{1}{2}\left[1 + \frac{2(K_C - K_D)}{3K_C + 2K_D}\left(\frac{a}{r}\right)^5\right]\mathbf{rr} : \nabla\nabla\langle T_C\rangle + \frac{5K_D}{5K_D + K_C}(T_D - \langle T_C\rangle)\frac{a}{r}, \quad [7.4]$$

where $\mathbf{r}$ is the position vector relative to the particle center. With [7.4] we now calculate

$$\mathscr{A}[\mathbf{q}_C] = \frac{15}{a^2}\frac{K_C K_D}{5K_D + K_C}(\bar{T}_D - \langle T_C\rangle), \quad [7.5]$$

$$\mathscr{T}[\mathbf{q}_C] = 2K_C\frac{K_C - K_D}{K_D + 2K_C}\nabla\langle T_C\rangle, \quad [7.6]$$

$$\mathscr{S}[\mathbf{q}_C] = \frac{5}{2}\frac{K_C K_D}{5K_D + K_C}(\langle T_C\rangle - \bar{T}_D)\mathbf{I}, \quad [7.7]$$

while $\mathscr{R}[\mathbf{q}_C]$ gives a contribution of higher order in $a/L$. Similarly, we have

$$\mathscr{A}[T_C] = \frac{K_C - K_D}{K_D + 2K_C}\nabla\langle T_C\rangle, \quad [7.8]$$

$$\mathscr{T}[T_C] = \frac{5K_D}{5K_D + K_C}(\bar{T}_D - \langle T_C\rangle)\mathbf{I}, \quad [7.9]$$

while $\mathscr{S}[T_C]$ is negligibly small. From [6.6] we then have

$$\langle \mathbf{q}_C\rangle = -\left(1 + \frac{K_D - K_C}{K_D + 2K_C}\epsilon_D\right)K_C\nabla\langle T_C\rangle - \frac{5K_D K_C}{5K_D + K_C}\nabla[\epsilon_D(\bar{T}_D - \langle T_C\rangle)], \quad [7.10]$$

so that the averaged conduction equation becomes

$$\rho_C C_{pC}\epsilon_C\frac{\partial\langle T_C\rangle}{\partial t} = \epsilon_C K_C\nabla\cdot\left[\left(1 + \frac{K_D - K_C}{K_D + 2K_C}\epsilon_D\right)\nabla\langle T_C\rangle\right]$$

$$+ 2K_C\nabla\cdot\left[\epsilon_D\frac{K_D - K_C}{K_D + 2K_C}\nabla\langle T_C\rangle\right] + \frac{15K_C K_D}{5K_D + K_C}\frac{\epsilon_D}{a^2}(\bar{T}_D - \langle T_C\rangle)$$

$$+ \frac{3}{2}\frac{5K_D K_C}{5K_D + K_C}\nabla^2[\epsilon_D(\bar{T}_D - \langle T_C\rangle)]. \quad [7.11]$$

To the same order of accuracy the second term in the right-hand side can be multiplied by $\epsilon_C$ to find

$$\epsilon_C\rho_C C_{pC}\frac{\partial\langle T_C\rangle}{\partial t} = \epsilon_C\nabla\cdot(K^*\nabla\langle T_C\rangle) + \frac{15}{a^2}\frac{K_C K_D}{5K_D + K_C}(1 + \tfrac{1}{2}a^2\nabla^2)[\epsilon_D(\bar{T}_D - \langle T_C\rangle)], \quad [7.12]$$

where

$$\frac{K^*}{K_C} = 1 + 3\epsilon_D \frac{K_D - K_C}{K_D + 2K_C} \tag{7.13}$$

is the effective thermal conductivity (see, e.g. Jeffrey 1973; Batchelor 1974).

From [6.13] the equation for the mean particle temperature is

$$nv\rho_D C_{pD} \frac{\partial \bar{T}_D}{\partial t} = \epsilon_D K_C \nabla^2 \langle T_C \rangle - \frac{15 K_C K_D}{5K_D + K_C} \frac{\epsilon_D}{a^2} (\bar{T}_D - \langle T_C \rangle) + \epsilon_D \bar{s}_D. \tag{7.14}$$

The total heat flux $\mathbf{q}_m$ in the mixture is given by

$$\mathbf{q}_m = \langle \mathbf{q}_C \rangle + \epsilon_D \mathscr{L}[\mathbf{q}_C] - \boldsymbol{\Sigma}_q, \tag{7.15}$$

with $\boldsymbol{\Sigma}_q$ defined in [A.14]. In the present quasi-steady case with a spatially uniform source the $\boldsymbol{\Sigma}_q$ term vanishes so that, using the previous results, we find

$$\mathbf{q}_m = -K^* \nabla \langle T_C \rangle. \tag{7.16}$$

*7.2. Convection*

In the case of convection, neglect of the time derivative in [7.1] and in the corresponding equation for the particles requires not only that $\langle T_C \rangle$ vary slowly, but also that the temperature of the surrounding fluid, as 'seen' by the particles, be slowly varying as well. This condition implies that the Péclet number Pe based on the particle–fluid relative velocity,

$$\mathrm{Pe} = \frac{2a|\tilde{\mathbf{w}} - \langle \mathbf{u}_C \rangle|}{D_C}, \tag{7.17}$$

be sufficiently small. If the Prandtl number is not too large, this condition is satisfied when the Reynolds number is small.

With this assumption, a closure for the convection problem requires the solution of the steady form of the convection–diffusion equation [7.1] subject to the condition [7.3] at large distance from the particle. As shown by Acrivos and Taylor (1962), this is a singular perturbation problem whose solution is not available at present. However, Acrivos and Taylor have presented a solution subject to the conditions of uniform flow at infinity and uniform temperature on the particle surface and at infinity. In order to use this solution we must relax the conditions [5.3] and [7.3] and impose instead

$$\langle T_C \rangle_1(\mathbf{z} \mid \mathbf{x}, \mathbf{w}) \to \langle T_C \rangle(\mathbf{x}), \tag{7.18}$$

$$\langle \mathbf{u}_C \rangle_1(\mathbf{z} \mid \mathbf{x}, \mathbf{w}) \to \langle \mathbf{u}_C \rangle(\mathbf{x}). \tag{7.19}$$

As a consequence, the averaged equations that we obtain have an error of order $a/L$ rather than $o(a/L)$. Batchelor (1979) and others have considered the heat or mass transfer from particles in non-uniform Stokes flow, but still with a uniform temperature at infinity.

In the present notation, near the particle, the solution of Acrivos and Taylor is

$$\langle T_C \rangle_1 = \langle T_C \rangle(\mathbf{x}) + (T_D - \langle T_C \rangle)\left\{\frac{a}{r} + \frac{\mathrm{Pe}}{2}\left[\frac{1}{2}\left(-1 + \frac{a}{r}\right) + \left(\frac{1}{2} - \frac{3a}{4r} + \frac{3a^2}{8r^2} - \frac{a^3}{8r^3}\right)\cos\theta\right]\right\}, \tag{7.20}$$

where the particle temperature $T_D$ has been assumed spatially uniform inside each particle.

With the temperature field [7.20] we find $\mathscr{A}[T_C] = 0$ and

$$\mathscr{T}[T_C] = (\bar{T}_D - \langle T_C \rangle)\mathbf{I}\left[1 + O\left(\frac{a}{L}\right)\right], \tag{7.21}$$

so that, from [6.6],

$$\langle \mathbf{q}_C \rangle = -K_C \nabla \langle T_C \rangle - K_C \nabla[\epsilon_D(\bar{T}_D - \langle T_C \rangle)]\left[1 + O\left(\frac{a}{L}\right)\right]. \tag{7.22}$$

We also have

$$\mathscr{A}[\mathbf{q}_C] = \frac{3K_C}{a^2}\left(1 + \frac{\mathrm{Pe}}{4}\right)(\bar{T}_D - \langle T_C\rangle)\left[1 + O\left(\frac{a}{L}\right)\right], \tag{7.23}$$

$$\mathscr{T}[\mathbf{q}_C] = \tfrac{3}{8}[\rho_C C_{pC}(\bar{T}_D - \langle T_C\rangle)(\bar{\mathbf{w}} - \langle \mathbf{u}_C\rangle) - \mathbf{Q}_D]\left[1 + O\left(\frac{a}{L}\right)\right], \tag{7.24}$$

$$\mathscr{S}[\mathbf{q}_C] = -\tfrac{1}{8}(4 + \mathrm{Pe})K_C(\bar{T}_D - \langle T_C\rangle)\mathbf{I}. \tag{7.25}$$

Actually, in view of the accuracy with which $\mathscr{A}[\mathbf{q}_C]$ is known, most of these terms must be dropped for consistency. For example, the second term of $\langle \mathbf{q}_C\rangle$ gives a contribution $K_C\nabla^2[\epsilon_D(\bar{T}_D - \langle T_C\rangle)] \sim O(a^3/L^2)$, which is clearly smaller than the first neglected term in $\mathscr{A}[\mathbf{q}_C]$. By a similar argument we see that all the terms in $\mathscr{T}[\mathbf{q}_C]$ and $\mathscr{S}[\mathbf{q}_C]$ must be dropped. The reason we have shown them is that these are the correct leading contributions to these quantities exhibiting some interesting physics. For example, $\mathscr{T}[\mathbf{q}_C]$ is the convective transport due to the relative velocity of the phases. Qualitatively, its origin lies in the same physical process that, at the molecular level, is responsible for ordinary conduction in a gas. In the present case of a dilute suspension of spherical particles the term $\mathscr{R}[\mathbf{q}_C]$ would not contribute even if $\mathscr{A}[\mathbf{q}_C]$ were known correctly to $O(a/L)$. However, it would not remain negligible for non-spherical particles or at higher concentrations.

As for $\mathbf{Q}_C$, the comments made earlier about $\mathbf{M}_C$ are also applicable. In particular, one may expect this quantity to be negligible whenever $\mathbf{M}_C$ is provided the Prandtl number is not too large.

Substitution of the results into [6.3] leads to the average energy equation for the continuous phase

$$\epsilon_C\left(\frac{\partial\langle T_C\rangle}{\partial t} + \langle \mathbf{u}_C\rangle\cdot\nabla\langle T_C\rangle\right) = \epsilon_C D_C\nabla^2\langle T_C\rangle + \frac{3\epsilon_D D_C}{a^2}\left(1 + \frac{\mathrm{Pe}}{4}\right)(\bar{T}_D - \langle T_C\rangle) + \nabla\cdot(\epsilon_C\mathbf{Q}_C). \tag{7.26}$$

The only difference with the conduction–diffusion equation of a pure medium is the next-to-the-last term that is analogous to the Stokes drag contribution in the momentum equation [5.7]. It is readily shown that its form is compatible with a Nusselt number for the particles given by $2 + \frac{1}{2}\mathrm{Pe}$ (Acrivos and Taylor 1962). The equation for the disperse phase is

$$nv\rho_D C_{pD}\left(\frac{\partial\bar{T}_D}{\partial t} + \bar{\mathbf{w}}\cdot\nabla\bar{T}_D\right) = \epsilon_D K_C\nabla^2\langle T_C\rangle + \epsilon_D\bar{s}_D + \nabla\cdot(\epsilon_D\mathbf{Q}_D)$$

$$- \frac{3\epsilon_D}{a^2}K_C(1 + \tfrac{1}{4}\mathrm{Pe})(\bar{T}_D - \langle T_C\rangle). \tag{7.27}$$

Note that, upon adding these two equations, one finds a heat flux in the mixture given by $-K_C\nabla\langle T_C\rangle$ rather than $-K^*\nabla\langle T_C\rangle$. The difference between these two quantities is comparable in magnitude with other terms that have not been included for the reasons previously discussed. Note also that, as shown by Acrivos *et al.* (1980), the limits of small volume fraction and small Péclet number cannot be interchanged. The calculation presented here corresponds to taking the limit $\epsilon_D \to 0$ prior to $\mathrm{Pe} \to 0$.

## 8. DISCUSSION

In order to compare the present results with the 'particle stress' introduced by Batchelor (1970), it is useful to derive equations for the total mixture momentum and energy. These equations of course are also of interest in themselves as their use is sometimes advocated in place of the two-fluid description given before.

In principle, the total mixture momentum balance is obtained by adding to [4.8] for $\langle \mathbf{u}_c \rangle$ and [A9] for $\langle \mathbf{u}_D \rangle$ rather than [4.12] for $\bar{\mathbf{w}}$ as, for the reasons already explained, $\rho_C \epsilon_C \langle \mathbf{u}_C \rangle$ and $\rho_D \epsilon_D \langle \mathbf{u}_D \rangle$ are the momentum densities of the phases in relation to the same unit volume.

Let $\rho_m$ and $\mathbf{U}$ denote the mixture density and center-of-mass velocity defined by

$$\rho_m = \epsilon_C \rho_C + \epsilon_D \rho_D, \quad \rho_m \mathbf{U} = \epsilon_C \rho_C \langle \mathbf{u}_C \rangle + \epsilon_D \rho_D \langle \mathbf{u}_D \rangle. \tag{8.1}$$

Then, upon adding [4.8] and [A9] it is easy to find

$$\frac{\partial}{\partial t}(\rho_m \mathbf{U}) + \nabla \cdot (\rho_m \mathbf{U}\mathbf{U}) = -\nabla(\epsilon_C \langle p_C \rangle) + 2\mu_C \nabla \cdot \mathbf{E}_m + \nabla \cdot (\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2 + \boldsymbol{\Sigma}_R + \boldsymbol{\Sigma}_a) + \rho_m \mathbf{g}. \tag{8.2}$$

In deriving this expression we have assumed a (microscopic) stress-strain relation of the Newtonian type for the continuous phase. The mean strain field $\mathbf{E}_m$ is defined in [3.10] and $\boldsymbol{\Sigma}_a$ in [A6]. Furthermore, we have used the definitions

$$\boldsymbol{\Sigma}_1 = \int d^3w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{z}-\mathbf{x}|=a} dS_z [a(\langle \boldsymbol{\sigma}_C \rangle_1 \cdot \mathbf{n})\mathbf{n} - \mu_C(\langle \mathbf{u}_C \rangle_1 \mathbf{n} + \mathbf{n}\langle \mathbf{u}_C \rangle_1)], \tag{8.3}$$

$$\boldsymbol{\Sigma}_2 = \nabla \cdot (\epsilon_D \mathscr{S}[\boldsymbol{\sigma}_C]) + \nabla\nabla : (\epsilon_D \mathscr{R}[\boldsymbol{\sigma}_C]) + 2\mu_C \,\mathrm{Symm}\{\nabla \cdot [\epsilon_D \mathscr{T}[\mathbf{u}_C]] - \nabla[\epsilon_D(\bar{\mathbf{w}} - \langle \mathbf{u}_C \rangle)] + \nabla\nabla : [\epsilon_D \mathscr{S}[\mathbf{u}_C]]\}, \tag{8.4}$$

$$\boldsymbol{\Sigma}_R = \rho_C \epsilon_C \mathbf{M}_C + \rho_D \epsilon_D (\langle \mathbf{u}_D \rangle \langle \mathbf{u}_D \rangle - \langle \mathbf{u}_D \mathbf{u}_D \rangle) - \frac{\rho_D \rho_C \epsilon_C \epsilon_D}{\rho_m} (\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle)(\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle). \tag{8.5}$$

The stress terms denoted by the $\boldsymbol{\Sigma}$'s may be compared with the expression for the 'particle stress' $\boldsymbol{\Sigma}^B$ introduced by Batchelor (1970). His original ensemble-average definition is expressed in terms of volume averages with the assumption of a uniform mixture. Upon restoring the ensemble averaging and setting $\boldsymbol{\Sigma}^B = \boldsymbol{\Sigma}_1^B + \boldsymbol{\Sigma}_2^B + \boldsymbol{\Sigma}_3^B$, his definition [4.5] in his paper is, in the present notation,

$$\boldsymbol{\Sigma}_1^B = \frac{1}{(N-1)!} \int d^3w^{(1)} \int_{|\mathbf{x}-\mathbf{z}|=a} dS_z^{(1)} \int d\mathscr{C}^{N-1} P(N; t)[a(\boldsymbol{\sigma}_C \cdot \mathbf{n})\mathbf{n} - \mu_C(\mathbf{u}_C \mathbf{n} + \mathbf{n}\mathbf{u}_C)]$$

$$= \int d^3w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{z}-\mathbf{x}|=a} dS_z [a(\langle \boldsymbol{\sigma}_C \rangle_1 \cdot \mathbf{n})\mathbf{n} - \mu_C(\langle \mathbf{u}_C \rangle_1 \mathbf{n} + \mathbf{n}\langle \mathbf{u}_C \rangle_1)], \tag{8.6}$$

$$\boldsymbol{\Sigma}_2^B = -\frac{1}{N!} \int d\mathscr{C}^N \mathbf{u}'\mathbf{u}' P(N; t), \tag{8.7}$$

$$\boldsymbol{\Sigma}_3^B = -\frac{1}{(N-1)!} \rho_D \int d^3w^{(1)} \int_{|\mathbf{x}-\mathbf{z}| \leq a} d^3z \int d\mathscr{C}^{N-1} P(N; t)\mathbf{f}'(\mathbf{z} - \mathbf{x}), \tag{8.8}$$

where $\mathbf{u}' = \chi_C \mathbf{u}_C + \chi_D \mathbf{u}_D - \mathbf{U}$ and $\mathbf{f}'$ is the difference between the local and the average acceleration of the mixture. The term $\boldsymbol{\Sigma}_1^B$ is manifestly identical to our $\boldsymbol{\Sigma}_1$.† Similarly, $\boldsymbol{\Sigma}_2^B$ is readily shown to be the same as $\boldsymbol{\Sigma}_R$. Furthermore, since $\mathbf{f}'$ and our $\mathbf{a}_D$ appearing in the definition [A6] of $\boldsymbol{\Sigma}_a$ differ by a constant vector which contributes nothing to integral in [8.8], it is evident that $\boldsymbol{\Sigma}_3^B$ equals our $\boldsymbol{\Sigma}_a$.

Due to his assumption of uniformity, Batchelor's expression does not contain our term $\boldsymbol{\Sigma}_2$. Actually, its neglect introduces a certain inconsistency for the following reason. From the definition [8.6] we can write, in order of magnitude,

$$\boldsymbol{\Sigma}_1 \sim \mu_C \langle \mathbf{u}_C \rangle \frac{\epsilon_D}{a}\left[1 + O\left(\frac{a}{L}\right)\right] \tag{8.9}$$

†To achieve this equivalence, the pressure term in [8.2] has been written as $\nabla(\epsilon_C \langle p_C \rangle)$ rather than in the perhaps more natural form $\nabla \langle p_C \rangle$ with the extra $\epsilon_D \langle p_C \rangle$ incorporated in the definition of $\boldsymbol{\Sigma}_1$.

while

$$\nabla \cdot (\epsilon_D \mathscr{S}[\boldsymbol{\sigma}_C]) \sim \mu_C \langle \mathbf{u}_C \rangle \frac{\epsilon_D}{L}. \qquad [8.10]$$

From these estimates, it would seem that $\boldsymbol{\Sigma}_I$ is dominant over the other term which can therefore legitimately be dropped. However, it is easy to convince oneself that, for spheres in Stokes flow, one cannot form a second-order tensor with the quantities shown in the right-hand side of [8.9] so that the leading order term of $\boldsymbol{\Sigma}_I$ must vanish. This conclusion is borne out by the explicit results for the dilute case. Therefore, $\boldsymbol{\Sigma}_I$ and $\nabla \cdot (\epsilon_D \mathscr{S}[\boldsymbol{\sigma}_C])$ *are of the same order*. For example, again in the dilute case, we can write the total momentum equation [8.2] using the results of appendix B to find

$$\frac{\partial}{\partial t}(\rho_m \mathbf{U}) + \nabla \cdot (\rho_m \mathbf{U}\mathbf{U}) = -\nabla \langle p_C \rangle + \nabla \cdot \boldsymbol{\Sigma}_v \qquad [8.11]$$

where the viscous stress tensor in the mixture $\boldsymbol{\Sigma}_v$ is given by

$$\boldsymbol{\Sigma}_v = 2\mu^* \mathbf{E}_m + \frac{\mu_C^2 \epsilon_D}{2(\mu_C + \mu_D)} \nabla \cdot [\epsilon_D(\bar{\mathbf{w}} - \langle \mathbf{u}_C \rangle)]\mathbf{I} + \frac{3}{4} \frac{\mu_C \mu_D}{\mu_C + \mu_D} \nabla[\epsilon_D(\bar{\mathbf{w}} - \langle \mathbf{u}_C \rangle)]. \qquad [8.12]$$

While the $\mu^*$ correction comes from $\boldsymbol{\Sigma}_I$, the other two terms are the divergence of $\epsilon_D \mathscr{S}[\boldsymbol{\sigma}_C]$. In situations such that $\langle \mathbf{u}_C \rangle$ and $(\bar{\mathbf{w}} - \langle \mathbf{u}_C \rangle)$ are comparable, all the terms in this relation have the same order of magnitude in $a/L$. In this case, the suspension will exhibit a non-Newtonian character even for rigid particles for which $\mu_C/\mu_D \to 0$. Of course this aspect would be unimportant if, as in colloids, $\bar{\mathbf{w}} \simeq \langle \mathbf{u}_C \rangle$.

The peculiar fact that the assumption of local uniformity gives the correct effective viscosity is a consequence of the tensorial nature of $\mathscr{S}[\boldsymbol{\sigma}_C]$ and Galilean invariance. Indeed, from the definition [2.18], the leading order term of $\mathscr{S}[\boldsymbol{\sigma}_C]$ (i.e. the term of order 0 in $a/L$) cannot involve gradients of averaged quantities and can only depend on velocities plus scalars (volume fractions, etc.). The only third-order tensor that can be formed with these quantities is therefore the outer product of a velocity and the identity tensor (cf. [5.6] in the dilute limit). On the other hand, since $\mathscr{S}[\boldsymbol{\sigma}_C]$ is Galilean invariant, it cannot depend on $\mathbf{u}_m$ but can only depend on the relative velocity. This argument shows therefore that the divergence of this term cannot give a contribute proportional to $\mathbf{E}_m$.

The argument given above to prove the vanishing of the dominant contribution of $\boldsymbol{\Sigma}_I$ is not applicable at finite particle Reynolds numbers because a quantity with the correct tensorial nature can be formed with the tensorial product of two velocities as showed in our earlier paper on potential flow (Zhang and Prosperetti 1994). In such a situation, therefore, the effective viscosity correction, which arises from the second term of [8.9], would be of smaller order than the dominant contribution of this term.

While the interphase drag term $\mathscr{A}[\boldsymbol{\sigma}_C]$ cancels upon combining the two momentum equations, it would certainly remain in the other equation needed to close the system, be that one for the relative motion or for the motion of one of the phases. Arguments similar to those given above indicate that this quantity cannot contain a term of order $a/L$ at any concentration. The only allowable leading order correction is of the Faxén type and gives a term of order $a^2/L^2$ as found in the dilute limit, [5.4]. This correction is therefore of the same order as the effective viscosity one. These comments show that a correction to the viscosity is only one of several aspects in which the flow of a suspension differs from that of a homogeneous fluid.

It may also be pointed out that, while the left-hand side of [8.2] is phrased in terms of the center-of-mass velocity $\mathbf{U}$, the Newtonian part of the stress tensor features the strain field of the volumetric velocity $\mathbf{u}_m$. The two coincide in the case of equal phase densities considered by Batchelor.

We can also compare the results of section 4.1 for the continuous-phase momentum equation with those of Joseph and Lundgren (1990). For this purpose we use [4.4] to rewrite the stress terms in the right-hand side of [4.8] in the form

$$\begin{aligned} \text{RHS} &\equiv \epsilon_C \nabla \cdot \langle \boldsymbol{\sigma}_C \rangle - \epsilon_D \mathscr{A}[\boldsymbol{\sigma}_C] + \nabla \cdot (\epsilon_D \mathscr{L}[\boldsymbol{\sigma}_C]) \\ &= \nabla \cdot (\epsilon_C \langle \boldsymbol{\sigma}_C \rangle) + \nabla \cdot [\epsilon_D \langle \boldsymbol{\sigma}_C \rangle + \mathscr{L}[\boldsymbol{\sigma}_C])]. \end{aligned} \qquad [8.13]$$

By expanding the terms $\langle \boldsymbol{\sigma}_C \rangle$ in the definition of $\mathscr{L}[\boldsymbol{\sigma}_C]$ up to $O(a/L)$ included, we readily find

$$\text{RHS} = \nabla \cdot (\epsilon_C \langle \boldsymbol{\sigma}_C \rangle) + \int d^3w \int_{|\mathbf{x}-\mathbf{y}|=a} dS_y P(\mathbf{y}, \mathbf{w}; t) \mathbf{n} \langle \boldsymbol{\sigma}_C \rangle_1(\mathbf{x}, t \mid \mathbf{y}, \mathbf{w}) \cdot \mathbf{n}. \quad [8.14]$$

The integral is identical to what Joseph and Lundgren denote by $\langle \delta_\Sigma(\mathbf{x})\mathbf{t} \rangle$. However, it is difficult to evaluate it because the integration is over all the particles that touch the point $\mathbf{x}$. The translation that we have effected in section 2 simplifies this task considerably. As for the other term in [8.14], for a Newtonian fluid, we recall [3.9] and find

$$\epsilon_C \langle \boldsymbol{\sigma}_C \rangle = -\epsilon_C \langle p_C \rangle \mathbf{I} + \mu_C [\nabla \mathbf{u}_m + (\nabla \mathbf{u}_m)^T], \quad [8.15]$$

which is exactly Joseph and Lundgren's result. As these authors have remarked, the difference between this result and that proposed by Ishii (1975, p. 165) and others is the difference between

$$\text{Symm}\{\nabla[\epsilon_D(\bar{\mathbf{w}} - \langle \mathbf{u}_C \rangle)]\}. \quad [8.16]$$

and

$$\text{Symm}\{(\bar{\mathbf{w}} - \langle \mathbf{u}_C \rangle)\nabla \epsilon_D\}. \quad [8.17]$$

While a considerable effort has been devoted to the analysis of the momentum equation, not much seems to have been done for the energy equation. The rather general results of Drew and Lahey (1993) are typical. Here we have explicitly closed the equations in some dilute-limit situations and we have given a formulation of greater potential usefulness for numerical simulation as will be discussed in section 9. The only detailed study of a problem in the heat transfer area is a paper by Acrivos *et al.* (1980). These authors studied the convection heat transfer from a fixed particle bed at small Péclet numbers carrying the calculation to second order in $\epsilon_D$. Their interest lay however more in the calculation of the average bed temperature than in the explicit closure of the equations.

A mixture energy equation can be derived by adding [6.5] and[A13] to find

$$C_{pC}\rho_C\left[\frac{\partial(\epsilon_C\langle T_C\rangle)}{\partial t} + \nabla \cdot (\epsilon_C \langle T_C \mathbf{u}_C \rangle)\right] + \rho_D C_{pD}\left[\frac{\partial \epsilon_D \langle T_D \rangle}{\partial t} + \nabla \cdot (\epsilon_D \langle \mathbf{u}_D T_D \rangle)\right]$$

$$= \nabla \cdot (K_C \nabla T_m) - \nabla \cdot (\mathbf{H}_1 + \mathbf{H}_2 - \boldsymbol{\Sigma}_q) + \epsilon_D \langle s_D \rangle. \quad [8.18]$$

Here, $\boldsymbol{\Sigma}_q$ is defined by [A14], $T_m$ is the mixture temperature defined by

$$T_m = \epsilon_C \langle T_C \rangle + \epsilon_D \langle T_D \rangle, \quad [8.19]$$

and

$$\mathbf{H}_1 = \int d^3w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{z}-\mathbf{x}|=a} dS_z [a(\langle \mathbf{q}_C \rangle_1 \cdot \mathbf{n})\mathbf{n} + K_C \langle T_C \rangle_1 \mathbf{n}], \quad [8.20]$$

$$\mathbf{H}_2 = \nabla \cdot (\epsilon_D \mathscr{S}[\mathbf{q}_C]) + \nabla\nabla : (\epsilon_D \mathscr{R}[\mathbf{q}_C])$$

$$- K_C\{\nabla \cdot (\epsilon_D \mathscr{T}[T_C]) - \nabla[\epsilon_D(\langle T_D \rangle - \langle T_C \rangle)] + \nabla\nabla : (\epsilon_D \mathscr{S}[T_C])\}. \quad [8.21]$$

The combination $\mathbf{H}_1 + \mathbf{H}_2 - \boldsymbol{\Sigma}_q$ plays the role of 'particle heat flux' analogous to Batchelor's particle stress. The structure of these terms is quite comparable to the corresponding ones in the momentum equation and many of the previous remarks apply to this case as well. In particular, in the presence of convection, the description of the thermal behavior of the mixture will require other terms in addition to an effective thermal conductivity. Examples are given in [7.24] and [7.25].

## 9. CONCLUSIONS

We have presented a general formulation for the average momentum and energy equations in a suspension of spherical particles in a continuous fluid phase. The main points of the paper are:

(1) Unlike several other studies that have focused on the calculation of the effective viscosity and other effective properties, we have addressed the derivation of averaged equations. The two approaches must be considered as complementary. The first one provides insight into the differences between specific situations (e.g. a fixed bed versus a sedimenting suspension). The present method may instead indicate general features of the equations. An example is our result for dilute Stokes flow, where we have found non-Newtonian effects that are, in principle, of the same order or larger than the viscosity correction. (Since this term contains the difference between the phase velocities, its importance will be small in many situations where this difference is small.) In obtaining this result, the applicability of the present method to cases with significant spatial gradients of the averaged field has been essential.

(2) As all other averaging methods, our approach does not lead to a closed system of equations. At low volume fractions the closure is rather straightforward in a number of cases of which we have presented examples in this and previous papers. For the more interesting dense case, we believe that our approach offers the distinct advantage that the quantities that need to be determined, such as the mean force and the mean stresslet-torque per particle, are amenable to evaluation by direct numerical simulation. An example of a possible procedure to exploit numerical results to close the equations has been presented in Zhang and Prosperetti (1994a) for the linear potential problem. The idea is to start with an educated guess as to the possible nature of the closure relation. The unknown part of the closure relation is then reduced to unknown coefficients that are evaluated with the aid of direct numerical simulations. We are currently working on other applications of this idea. This approach is a promising one in view of the rapid progress in the development of techniques for the direct numerical simulation of the motion of particles in fluids (e.g. Durlofsky *et al.* 1987; Sulsky and Brackbill 1991; Sangani and Didwania 1993b; Claeys and Brady 1993; Cichocki *et al.* 1994; Ladd 1993; Feng *et al.* 1994).

(3) By averaging Newton's equation for the particles directly, we have bypassed the problems connected with the evaluation and interpretation of the stress term in the disperse phase momentum equation. This aspect of suspension modelling has caused difficulties in the past (see, e.g. Givler 1987; Drew and Lahey 1993; Hwang and Shen 1989). Actually, as shown in appendix A and also in Prosperetti and Zhang (1996), our procedure allows one to calculate this term easily and unambiguously.

(4) We have applied our methods also to the derivation of the average energy equations and we have presented explicit closures for some dilute cases.

(5) Our approach directly leads to equations in the 'two-fluid' form widely used in Engineering. Alternative formulations in terms of equations for the mixture and one of the phases follow immediately.

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# APPENDIX A

*Phase Averaging for the Disperse Phase*

In sections 4 and 6 we have derived the disperse phase momentum and energy equations by using the 'particle' averaging defined in [2.22]. Here we provide an alternative treatment in terms of the phase-averaged quantities defined in [2.8]. This approach offers the advantage that the balance equations for both phases refer to the same unit volume (cf. the comments following [2.29] and [4.10]). It could be constructed so as to closely parallel the analysis of the continuous-phase strain rate given in section 3 and to lead to a momentum equation in which the average acceleration of the disperse phase is determined by the average stress of the disperse phase. A disadvantage of such a formulation, however, is the need to consider in detail the particle constitutive relation even in cases in which a simple model, e.g. that of a rigid particle, should suffice. For this reason we follow a slightly different path in which the dynamic boundary condition at the particle surface is invoked early on to eliminate the stress tensor of the particle material. This leads to a considerable simplification of the formulation.

We start from the unaveraged momentum equation for the particle material (assumed incompressible for simplicity):

$$\rho_D \mathbf{a}_D \equiv \frac{\partial \rho_D \mathbf{u}_D}{\partial t} + \nabla \cdot (\rho_D \mathbf{u}_D \mathbf{u}_D) = \nabla \cdot \boldsymbol{\sigma}_D, \tag{A1}$$

where $\mathbf{a}_D$ and $\boldsymbol{\sigma}_D$ are the acceleration and stress of the disperse phase material.

After the application of a transport theorem similar to [2.20], namely

$$\frac{\partial \epsilon_D \langle f_D \rangle}{\partial t} + \nabla \cdot (\epsilon_D \langle f_D \mathbf{u}_D \rangle) = \epsilon_D \left\langle \frac{\partial f_D}{\partial t} + \nabla \cdot (f_D \mathbf{u}_D) \right\rangle, \tag{A2}$$

one finds

$$\frac{\partial}{\partial t} (\epsilon_D \rho_D \langle \mathbf{u}_D \rangle) + \nabla \cdot (\epsilon_D \rho_D \langle \mathbf{u}_D \mathbf{u}_D \rangle) = \epsilon_D \langle \nabla \cdot \boldsymbol{\sigma}_D \rangle + \epsilon_D \rho_D \mathbf{g}. \tag{A3}$$

From the definitions [2.8] and [2.10] one can easily show that (Zhang 1994)

$$\epsilon_D \langle \nabla \cdot \boldsymbol{\sigma}_D \rangle = \int_{|\mathbf{x}-\mathbf{y}| \leq a} \mathrm{d}^3 y \int \mathrm{d}^3 w P(\mathbf{y}, \mathbf{w}; t) \langle \nabla \cdot \boldsymbol{\sigma}_D \rangle_1 (\mathbf{x}, t \mid \mathbf{y}, \mathbf{w}). \tag{A4}$$

The integral here is over all the particles that contain the point $\mathbf{x}$ but, proceeding as in the case of [2.13], it can be reduced approximately to one over a fixed particle to find

$$\epsilon_D\langle\nabla\cdot\boldsymbol{\sigma}_D\rangle = \int d^3w P(\mathbf{x},\mathbf{w};t)\int_{|\mathbf{z}-\mathbf{x}|\le a} d^3z\langle\nabla_z\cdot\boldsymbol{\sigma}_D\rangle_1(\mathbf{z},t\mid\mathbf{x},\mathbf{w}) + \nabla\cdot\boldsymbol{\Sigma}_a + O\left(\frac{a^2}{L^2}\epsilon_D\langle\nabla\cdot\boldsymbol{\sigma}_D\rangle_1\right), \quad [A5]$$

where the second-order tensor $\boldsymbol{\Sigma}_a$ is given by

$$\boldsymbol{\Sigma}_a = -\int d^3w P(\mathbf{x},\mathbf{w})\int_{|\mathbf{z}-\mathbf{x}|\le a} d^3z\langle\nabla_z\cdot\boldsymbol{\sigma}_D\rangle_1(\mathbf{z},t\mid\mathbf{x},\mathbf{w})(\mathbf{z}-\mathbf{x})$$

$$= -\rho_D\int d^3w P(\mathbf{x},\mathbf{w})\int_{|\mathbf{z}-\mathbf{x}|\le a} d^3z\langle\mathbf{a}_D\rangle_1(\mathbf{z},t\mid\mathbf{x},\mathbf{w})(\mathbf{z}-\mathbf{x}). \quad [A6]$$

The first term in the right-hand side of [A5] can be further manipulated by interchanging the conditional averaging and the integration over the particle volume and then applying the divergence theorem to write it as an integral over the particle surface. Upon using the dynamic boundary condition

$$\boldsymbol{\sigma}_D\cdot\mathbf{n} = \boldsymbol{\sigma}_C\cdot\mathbf{n} + \mathbf{S}, \quad [A7]$$

where $\mathbf{S}$ denotes the surface force density, we then have

$$\int d^3w P(\mathbf{x},\mathbf{w};t)\int_{|\mathbf{z}-\mathbf{x}|\le a} d^3z\langle\nabla_z\cdot\boldsymbol{\sigma}_D\rangle_1 = \int d^3w P(\mathbf{x},\mathbf{w};t)\int_{|\mathbf{x}-\mathbf{z}|=a} dS_z\langle\boldsymbol{\sigma}_C\rangle_1\cdot\mathbf{n} + \int d^3w P(\mathbf{x},\mathbf{w};t)\int_{|\mathbf{x}-\mathbf{z}|=a} dS_z\mathbf{S}. \quad [A8]$$

The last term is the total surface force on the particle and vanishes as shown in Prosperetti and Jones (1984) and Hesla *et al.* (1993). Finally, by [4.11], we find the following form for the disperse phase momentum equation:

$$\rho_D\left[\frac{\partial\epsilon_D\langle\mathbf{u}_D\rangle}{\partial t} + \nabla\cdot(\epsilon_D\langle\mathbf{u}_D\mathbf{u}_D\rangle)\right] = \epsilon_D\nabla\cdot\langle\boldsymbol{\sigma}_C\rangle + \epsilon_D\mathscr{A}[\boldsymbol{\sigma}_C] + \nabla\cdot\boldsymbol{\Sigma}_a + \rho_D\epsilon_D\mathbf{g} + O\left(\frac{a^2}{L^2}\epsilon_D\langle\nabla\cdot\boldsymbol{\sigma}_D\rangle_1\right). \quad [A9]$$

It is interesting to compare this result with the one in terms of $\tilde{\mathbf{w}}$ given in section 4. To this end we use once again the transport theorem [A2] and find

$$\rho_D\left[\frac{\partial\epsilon_D\langle\mathbf{u}_D\rangle}{\partial t} + \nabla\cdot(\epsilon_D\langle\mathbf{u}_D\mathbf{u}_D\rangle)\right] = \int_{|\mathbf{x}-\mathbf{y}|\le a} d^3y\int d^3w P(\mathbf{y},\mathbf{w};t)\langle\rho_D\mathbf{a}_D\rangle_1(\mathbf{x},t\mid\mathbf{y},\mathbf{w}). \quad [A10]$$

We now effect a translation similar to [2.12] to obtain

$$\rho_D\left[\frac{\partial\epsilon_D\langle\mathbf{u}_D\rangle}{\partial t} + \nabla\cdot(\epsilon_D\langle\mathbf{u}_D\mathbf{u}_D\rangle)\right] = \int d^3w P(\mathbf{x},\mathbf{w};t)\left\langle\int_{|\mathbf{x}-\mathbf{z}|\le a} d^3z\rho_D\mathbf{a}_D(\mathbf{z},t\mid\mathbf{x},\mathbf{w})\right\rangle_1 + \nabla\cdot\boldsymbol{\Sigma}_a. \quad [A11]$$

The inner integral in the right-hand side obviously equals the particle mass times its center-of-mass acceleration $\dot{\mathbf{w}}$ so that we have

$$\rho_D\left[\frac{\partial \epsilon_D\langle \mathbf{u}_D\rangle}{\partial t} + \nabla\cdot(\epsilon_D\langle \mathbf{u}_D\mathbf{u}_D\rangle)\right] = nm\bar{\dot{\mathbf{w}}} + \nabla\cdot\boldsymbol{\Sigma}_a. \tag{A12}$$

Upon using this relation to express the left-hand side of [A9] one recovers [4.12]. This argument is consistent with the earlier statement that the description in terms of $\bar{\mathbf{w}}$ is not an approximation but is just as legitimate as the one in terms of $\langle \mathbf{u}_D\rangle$.

A procedure analogous to that leading to [A9] can be followed for the energy equation. The result is

$$\rho_D C_{pD}\left[\frac{\partial \epsilon_D\langle T_D\rangle}{\partial t} + \nabla\cdot(\epsilon_D\langle \mathbf{u}_D T_D\rangle)\right] = -\epsilon_D\nabla\cdot\langle \mathbf{q}_C\rangle - \epsilon_D\mathscr{A}[\mathbf{q}_C] + \nabla\cdot\boldsymbol{\Sigma}_q + \epsilon_D\langle s_D\rangle + O\left(\frac{a^2}{L^2}\epsilon_D\langle\nabla\cdot\mathbf{q}_D\rangle_I\right), \tag{A13}$$

where, as in [A6],

$$\boldsymbol{\Sigma}_q = -\int d^3w P(\mathbf{x},\mathbf{w})\int_{|\mathbf{z}-\mathbf{x}|\le a} d^3z(\mathbf{z}-\mathbf{x})\langle\nabla_z\cdot\mathbf{q}_D\rangle_I(\mathbf{z},t\mid\mathbf{x},\mathbf{w}). \tag{A14}$$

In conclusion we note a possible extension of the particle-average idea [2.22]. Even in cases in which the detailed internal behavior of the particles is important, one can describe it in terms of global particle quantities, rather than local fields. For instance, the flow inside a droplet may be broken up into an expansion in spherical harmonics rather than described by the point-wise values of the velocity and pressure fields. One can then write a dynamical 'particle-average' equation for the coefficients of each order of the expansion. The first equation of this family would be the $\bar{\mathbf{w}}$ momentum equation [4.12]. An example of this procedure was presented in Zhang and Prosperetti (1994b) where an evolution equation for the radius of spherical bubbles was derived. This comment is presented to point out that the description in terms of particle averages is not necessarily approximate, but can be made, in principle, as accurate as required. A practical advantage of this approach is that one is not forced to deal with the entire fields, but the analysis can be limited to the first few coefficients.

# APPENDIX B

*Suspensions of Drops*

We now give a dilute-limit closure comparable to that of section 5 for a suspension of spherical drops. For simplicity, we neglect the relative rotation between the drop fluid and the surrounding continuous phase.

In this case the solution of the problem requires a suitable matching of $\langle \mathbf{u}_C\rangle_I$ to the flow inside the drop. It is easy to show that, because of the linearity of the Stokes equations, to the accuracy required in $\epsilon_D$, $\langle \mathbf{u}_D\rangle_I$, $\langle p_D\rangle_I$ also satisfy [5.1], [5.2]. The boundary conditions are obtained by taking the conditional average of the usual statements of continuity of velocity and tangential stress with the jump in normal stress needed to accommodate the effect of surface tension. To the present order in $\epsilon_D$, these relations are formally identical to the unaveraged ones. As in the case of rigid particles, therefore, the problem for the conditionally averaged fields is formally the same as that for the flow around an isolated spherical drop in the ambient flow [5.3].

By using [4.4] and the Faxén theorem result for the force on a drop in an arbitrary ambient flow (see, e.g. Kim and Karrila 1991, p. 78), accurate to $O(1)$ in $\epsilon_D$, we find in place of [5.4]

$$\mathscr{A}[\boldsymbol{\sigma}_C] = -\frac{3}{a^2}\mu_C\frac{\mu_C+\frac{3}{2}\mu_D}{\mu_C+\mu_D}(\bar{\mathbf{w}}-\langle \mathbf{u}_C\rangle) + \frac{3}{4}\frac{\mu_D}{\mu_C+\mu_D}\mu_C\nabla^2\langle \mathbf{u}_C\rangle. \tag{B1}$$

Similarly, for $\mathscr{T}[\boldsymbol{\sigma}_C]$, we use [4.5] and find

$$\mathscr{T}[\boldsymbol{\sigma}_C] = 3\alpha\mu_C \mathbf{E}_C, \tag{B2}$$

where

$$\alpha = \frac{\mu_D + \frac{2}{3}\mu_C}{\mu_D + \mu_C}. \tag{B3}$$

As before, the last two terms in [5.3] give contributions of higher order in $a/L$ to $\mathscr{S}[\boldsymbol{\sigma}_C]$ and $\mathscr{R}[\boldsymbol{\sigma}_C]$ which are themselves higher-order corrections and can therefore be calculated by using the well-known solution for a drop immersed in a uniform flow (see, e.g. Batchelor 1967; Kim and Karrila 1991). The result is

$$\epsilon_D(\mathscr{S}[\boldsymbol{\sigma}_C])_{ijk} = \tfrac{3}{4}\mu_C \frac{\mu_D}{\mu_C + \mu_D}\epsilon_D(\bar{w}_i - \langle u_{Ci}\rangle)\delta_{jk} + \tfrac{3}{10}\epsilon_D \frac{\mu_C^2}{\mu_D + \mu_C}$$
$$\times [(\bar{w}_i - \langle u_{Ci}\rangle)\delta_{jk} + (\bar{w}_j - \langle u_{Cj}\rangle)\delta_{ik} + (\bar{w}_k - \langle u_{Ck}\rangle)\delta_{ij}]. \tag{B4}$$

Due to the particular symmetry of $\langle\boldsymbol{\sigma}_C\rangle_1$ for a spherical drop in the dilute limit, the contribution of $\mathscr{R}[\boldsymbol{\sigma}_C]$ is found to be $O(a^2/L^2)$ and can, therefore, be neglected.

We now need to calculate the stress tensor. Since $\nabla \cdot \langle\boldsymbol{\sigma}_C\rangle \sim \langle\boldsymbol{\sigma}_C\rangle/L$ appears in the equations, in order to calculate the average viscous stress we only need an accuracy of order $a/L$. It is thus permissible to drop the last term in the right-hand side of [5.3] so that we can use the known solution for a drop in a shear flow (see, e.g. Batchelor 1967; Kim and Karrila 1991). In this way, the following results for the terms in the right-hand side of [3.3] are found:

$$\text{Symm}\{\mathscr{A}[\mathbf{u}_C]\} = -\alpha\mathbf{E}_C, \tag{B5}$$

$$(\mathscr{T}[\mathbf{u}_C])_{ijk} = \frac{1}{2}\frac{\mu_C + 2\mu_D}{\mu_C + \mu_D}(\bar{w}_i - \langle u_{Ci}\rangle)\delta_{jk} + \frac{1}{10}\frac{\mu_C}{\mu_C + \mu_D}(\bar{w}_l - \langle u_{Cl}\rangle)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{B6}$$

$$\nabla \cdot (\epsilon_D \mathscr{S}[\mathbf{u}_C])\} = O\left(\frac{a^2}{L^2}\right) \simeq 0. \tag{B7}$$

Using [3.3], one then finds the average of the microscopic strain rate $\langle\mathbf{e}_C\rangle$ for the continuous phase as

$$\langle\mathbf{e}_C\rangle = \frac{1}{\epsilon_C}\mathbf{E}_m - \frac{3\mu_C\epsilon_D}{5(\mu_C + \mu_D)}[\mathbf{E}_C + \tfrac{1}{2}(\mathbf{Z}_C - \tfrac{1}{3}(\text{Tr}\mathbf{Z}_C)\mathbf{I})], \tag{B8}$$

where

$$\mathbf{Z}_C = \frac{1}{\epsilon_D}\text{Symm}\{\nabla[\epsilon_D(\bar{\mathbf{w}} - \langle\mathbf{u}_C\rangle)]\}, \tag{B9}$$

and $\mathbf{E}_m$ is the strain field of the mean volume velocity $\mathbf{u}_m$ defined in [3.10].

Since, as a consequence of [2.21] and [2.25], $\nabla \cdot \mathbf{u}_m = \mathbf{0}$, it follows that $\text{Tr}\langle\mathbf{e}_C\rangle = \langle\text{Tr}\,\mathbf{e}_C\rangle = 0$ in the rigid particle limit in which $\mu_C/\mu_D \to 0$. However it is readily verified that, for finite $\mu_D$, $\text{Tr}\langle\mathbf{e}_C\rangle$ as given by [B8] vanishes to $O(\epsilon_D)$ but not exactly. This fact is compatible with the present approximations, but it might be useful to deal with an exactly divergenceless form that may be obtained by adding to [B8] an $O(\epsilon_D)$ term. While this form may not be unique, [B8] suggests

$$\langle\mathbf{e}_C\rangle = \frac{1}{\epsilon_C}\mathbf{E}_m - \frac{3\mu_C\epsilon_D}{5(\mu_C + \mu_D)}[\mathbf{E}_m + \tfrac{1}{2}(\mathbf{Z}_C - \tfrac{1}{3}(\text{Tr}\,\mathbf{Z}_C)\mathbf{I})]. \tag{B10}$$

With the previous results, the momentum equations now become

$$\epsilon_C\rho_C\left(\frac{\partial\langle\mathbf{u}_C\rangle}{\partial t}+\langle\mathbf{u}_C\rangle\cdot\nabla\langle\mathbf{u}_C\rangle\right)=-\epsilon_C\nabla\langle p_C\rangle$$

$$+\epsilon_C\nabla\cdot\left[2\mu^*\mathbf{E}_m+\frac{\mu_C^2\epsilon_D}{2(\mu_C+\mu_D)}(\mathrm{Tr}\,\mathbf{Z}_C)\mathbf{I}\right]+\tfrac{3}{4}\mu_C\frac{\mu_D}{\mu_C+\mu_D}\nabla^2[\epsilon_D(\bar{\mathbf{w}}-\langle\mathbf{u}_C\rangle)]$$

$$+\frac{3\epsilon_D\mu_C}{a^2}\frac{\mu_C+\frac{3}{2}\mu_D}{\mu_C+\mu_D}(\bar{\mathbf{w}}-\langle\mathbf{u}_C\rangle)-\frac{3}{4}\frac{\mu_D}{\mu_C+\mu_D}\epsilon_D\mu_C\nabla^2\langle\mathbf{u}_C\rangle$$

$$+\nabla\cdot(\epsilon_C\mathbf{M}_C)+\epsilon_C\rho_C\mathbf{g}, \qquad [B11]$$

$$\epsilon_D\rho_D\left(\frac{\partial\bar{\mathbf{w}}}{\partial t}+\bar{\mathbf{w}}\cdot\nabla\bar{\mathbf{w}}\right)=-\epsilon_D\nabla\langle p_C\rangle+2\mu_C\epsilon_D\nabla\cdot\mathbf{E}_C$$

$$-\frac{3\mu_C\epsilon_D}{a^2}\frac{\mu_C+\frac{3}{2}\mu_D}{\mu_C+\mu_D}(\bar{\mathbf{w}}-\langle\mathbf{u}_C\rangle)+\frac{3}{4}\frac{\mu_D}{\mu_C+\mu_D}\mu_C\epsilon_D\nabla^2\langle\mathbf{u}_C\rangle$$

$$+\nabla\cdot(\epsilon_D\mathbf{M}_D)+\epsilon_D\rho_D\mathbf{g}, \qquad [B12]$$

where $\mathbf{Z}_C$, $\mathbf{E}_C$ are given by [B9], [3.2], and $\mu^*$ is the well-known effective viscosity of a dilute suspension of drops (see, e.g. Batchelor 1967)

$$\frac{\mu^*}{\mu_C}=1+\tfrac{5}{2}\alpha\epsilon_D+o(\epsilon_D). \qquad [B13]$$

These two equations reduce to the corresponding ones previously given in section 5 upon taking the limit $\mu_C/\mu_D\to 0$, as expected.

# APPENDIX C

*Particle Collision Stress*

In this appendix we prove that the mean force resulting from direct particle contact can be represented as the divergence of a suitable stress. The proof is actually applicable to other forms of direct particle interactions, such as electrostatic or other forces, but such generality is unnecessary here.

Let $\mathbf{f}_c^\alpha$ be the total force acting on particle $\alpha$ due to the direct interaction with other particles. We write

$$\mathbf{f}_c^\alpha=\sum_{\beta\neq\alpha}\mathbf{f}(\mathbf{y}^{(\alpha)},\mathbf{y}^{(\beta)},\mathbf{w}^{(\alpha)},\mathbf{w}^{(\beta)},\mathscr{C}^{N-2};t)\equiv\sum_{\beta\neq\alpha}\mathbf{f}(N;t), \qquad [C1]$$

where $\mathbf{f}(\mathbf{y}^{(\alpha)},\mathbf{y}^{(\beta)},\dots)$ is the direct force exerted by the particle centered at $\mathbf{y}^\beta$ on that centered at $\mathbf{y}^\alpha$. Clearly, this decomposition in no way implies a restriction to binary collisions, nor to short-duration interactions.

According to the definition of particle average, the average particle contact force can then be written as

$$n\bar{\mathbf{f}}_c=\frac{1}{(N-1)!}\sum_{\alpha\neq\beta}\int d^3w^{(\alpha)}\int d\mathscr{C}^{N-1}\mathbf{f}(N;t)P(N;t). \qquad [C2]$$

Since particles $\alpha$ and $\beta$ are indistinguishable, we have

$$P(\mathbf{y}^{(\alpha)},\mathbf{y}^{(\beta)},\mathbf{w}^{(\alpha)},\mathbf{w}^{(\beta)},\mathscr{C}^{N-2};t)=P(\mathbf{y}^{(\beta)},\mathbf{y}^{(\alpha)},\mathbf{w}^{(\beta)},\mathbf{w}^{(\alpha)},\mathscr{C}^{N-2};t), \qquad [C3]$$

and, from Newton's third law, one can write

$$\mathbf{f}(\mathbf{y}^{(\alpha)}, \mathbf{y}^{(\beta)}, \mathbf{w}^{(\alpha)}, \mathbf{w}^{(\beta)}, \mathscr{C}^{N-2}; t) = -\mathbf{f}(\mathbf{y}^{(\beta)}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\beta)}, \mathbf{w}^{(\alpha)}, \mathscr{C}^{N-2}; t), \tag{C4}$$

from which, if $(\mathbf{f}P)$ denotes the product of $\mathbf{f}$ and $P$, we also have

$$(\mathbf{f}P)(\mathbf{y}^{(\alpha)}, \mathbf{y}^{(\beta)}, \mathbf{w}^{(\alpha)}, \mathbf{w}^{(\beta)}, \mathscr{C}^{N-2}; t) = -(\mathbf{f}P)(\mathbf{y}^{(\beta)}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\beta)}, \mathbf{w}^{(\alpha)}, \mathscr{C}^{N-2}; t). \tag{C5}$$

We now make a change of variables from $\mathbf{y}^{(\beta)}, \mathbf{y}^{(\alpha)}, \ldots, \mathbf{y}^{(N)}$ to $\mathbf{y}^{(\beta)}, \mathbf{r}^{(\alpha)}, \ldots, \mathbf{r}^{(\gamma)}$, where $\mathbf{r}^{(\alpha)} = \mathbf{y}^{(\alpha)} - \mathbf{y}^{(\beta)}, \ldots, \mathbf{r}^{(N)} = \mathbf{y}^{(N)} - \mathbf{y}^{(\beta)}$ and write

$$-(\mathbf{f}P)(\mathbf{y}^{(\beta)}, \mathbf{y}^{(\alpha)}, \mathbf{w}^{(\beta)}, \mathbf{w}^{(\alpha)}, \mathscr{C}^{N-2}; t) = -(\mathbf{f}P)(\mathbf{y}^{(\beta)}, \mathbf{r}^{(\alpha)}, \mathbf{r}^{(\alpha)}, \mathscr{R}^{N-2}; t). \tag{C6}$$

Here we use $\mathscr{R}^{N-2}$ as short-hand for $\mathbf{w}^{\alpha}, \mathbf{w}^{\beta}$ and all the $(\mathbf{r}^{\gamma}, \mathbf{w}^{\gamma})$ with $\gamma \neq \alpha, \beta$.

Now we apply Taylor's theorem to the right-hand side of [C6] as done above for [2.9] and write

$$(\mathbf{f}P)(\mathbf{y}^{(\beta)}, \mathbf{r}^{(\alpha)}, \mathscr{R}^{N-2}; t) = (\mathbf{f}P)(\mathbf{y}^{(\alpha)}, \mathbf{r}^{(\alpha)}, \mathscr{R}^{N-2}; t) - \mathbf{r}^{(\alpha)} \cdot \nabla_{\mathbf{y}^{\alpha}}(\mathbf{f}P)(\mathbf{y}^{(\alpha)} + \mathbf{h}, \mathbf{r}^{(\alpha)}, \mathscr{R}^{N-2}; t), \tag{C7}$$

where $\mathbf{h} = \mathbf{h}(\mathbf{y}^{(\alpha)}, \mathbf{r}^{(\alpha)}, \mathscr{R}^{N-2}; t)$, and $|\mathbf{h}| < |\mathbf{r}^{(\alpha)}|$. This relation essentially states that the interaction force when the particles have certain positions equals that that would prevail if all the particles were rigidly translated by $-\mathbf{r}^{(\alpha)}$, leaving their velocities unchanged, plus a correction term.

Upon substitution of the right-hand side of [C7] into [C2] and change of the integration variables from $\mathbf{y}^{(\beta)}, \ldots, \mathbf{y}^{(N)}$ to $\mathbf{r}^{(\alpha)}, \ldots, \mathbf{r}^{(N)}$ we have

$$n\overline{\mathbf{f}}_c = -\frac{1}{(N-1)!} \sum_{\beta \neq \alpha} \int \mathrm{d}^3 r^{(\alpha)}\, \mathrm{d}\mathscr{R}^{N-2} (\mathbf{f}P)(\mathbf{y}^{(\alpha)}, \mathbf{r}^{(\alpha)}, \mathscr{R}^{N-2}; t)$$

$$+ \frac{1}{(N-1)!} \sum_{\beta \neq \alpha} \nabla_{\mathbf{y}^{(\alpha)}} \cdot \int \mathrm{d}^3 r^{(\alpha)}\, \mathrm{d}\mathscr{R}^{N-2} \mathbf{r}^{(\alpha)} (\mathbf{f}P)(\mathbf{y}^{(\alpha)} + \mathbf{h}, \mathbf{r}^{(\alpha)}, \mathscr{R}^{N-2}; t), \tag{C8}$$

where the vector index of the gradient is the same as that of $\mathbf{r}^{(\alpha)}$. Note that the integration over $\mathbf{w}^{(\alpha)}$ is included in $\mathrm{d}\mathscr{R}^{N-2}$, and that the dependence on the index $\beta$ is implicit in the definition of $\mathbf{r}^{(\alpha)}$.

By changing now the integration variables to

$$\mathbf{z}^{(\beta)} = \mathbf{y}^{(\alpha)} + \mathbf{r}^{(\alpha)}, \ldots, \mathbf{z}^{(\gamma)} = \mathbf{y}^{(\alpha)} + \mathbf{r}^{(\gamma)}, \quad \gamma \neq \alpha, \beta, \tag{C9}$$

we can rewrite [C8] as

$$n\overline{\mathbf{f}}_c = -\frac{1}{(N-1)!} \sum_{\alpha \neq \beta} \int \mathrm{d}^3 w^{(\alpha)}\, \mathrm{d}^3 z^{(\beta)}\, \mathrm{d}^3 w^{(\beta)} \int \mathrm{d}\mathscr{C}^{N-2} (\mathbf{f}P)(\mathbf{y}^{(\alpha)}, \mathbf{z}^{(\beta)}, \mathbf{w}^{(\alpha)}, \mathbf{w}^{(\beta)}, \mathscr{C}^{N-2}; t)$$

$$+ 2\nabla \cdot (\epsilon_D \boldsymbol{\sigma}_c), \tag{C10}$$

where $\boldsymbol{\sigma}_c$ is the particle–particle contact stress defined, after a renaming of the variables, by

$$(\epsilon_D \boldsymbol{\sigma}_c)_{ij} = \frac{1}{2(N-1)!} \sum_{\alpha \neq \beta} \int \mathrm{d}^3 w^{(\alpha)}\, \mathrm{d}^3 y^{(\beta)}\, \mathrm{d}^3 w^{(\beta)} \int \mathrm{d}\mathscr{C}^{N-2}$$

$$\times (\mathbf{y}^{(\beta)} - \mathbf{y}^{(\alpha)})_j (\mathbf{f}P)_i (\mathbf{y}^{(\alpha)} + \mathbf{h}, \mathbf{y}^{(\beta)}, \mathbf{w}^{(\alpha)}, \mathbf{w}^{(\beta)}, \mathscr{C}^{N-2}; t). \tag{C11}$$

Other than for the name of the integration variables, the integral in [C10] is identical with that in the definition [C2] of $n\overline{\mathbf{f}}_c$, so that we may write

$$n\overline{\mathbf{f}}_c = -n\overline{\mathbf{f}}_c + 2\nabla \cdot (\epsilon_D \boldsymbol{\sigma}_c), \tag{C12}$$

from which

$$n\overline{\mathbf{f}}_c = \nabla \cdot (\epsilon_D \boldsymbol{\sigma}_c). \tag{C13}$$

In this way we have rigorously proven that the average force resulting from direct particle contact can be written as the divergence of a stress tensor. This proof generalizes a similar one in kinetic theories of molecular transport and rapid granular flows, where binary collisions are assumed (Lun *et al.* 1984). Sangani and Didwania (1993a) have given a similar proof for binary collisions of spheres in an inviscid fluid. In the proof presented here, the assumptions of binary, or very brief, contacts are unnecessary. These features are particularly important in the study of the rheological properties of a dense collection of particles. Based on [C11], a visco-elastic model for the slow flow of resin-coated sand has been developed (Zhang and Rauenzahn 1997).

If the maximum distance for particle–particle contact is much smaller than the macroscopic length scale, the term $\mathbf{h}$ in the definition of $\boldsymbol{\sigma}_c$ can approximately be set to zero, which amounts to retaining only the first correction in the Taylor series expansion of [C7]. This procedure is standard in all previous derivations. A detailed study of the consequences of dropping higher order terms has not yet been carried out.

In order to compare our result for $\boldsymbol{\sigma}_c$ with the corresponding one in the theory of rapid granular flow, we rewrite the expression [C11] in terms of conditional averages with two particles fixed:

$$\epsilon_D(\boldsymbol{\sigma}_c)_{ij} = \frac{1}{2}\int \mathrm{d}^3w^{(1)}\,\mathrm{d}^3w^{(2)}\,\mathrm{d}^3y^{(2)}(y_j^{(2)} - y_j^{(1)})\overline{f}_i^{(2)}(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}; t)P(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}; t). \quad [C14]$$

For the case of binary collisions of smooth, slightly inelastic particles, it is easy to see that the conditional force can be calculated as

$$\overline{\mathbf{f}}^{(2)} = -\tfrac{1}{2}(1 + e)m(\mathbf{q}\cdot\mathbf{n})^2\mathbf{n}\delta(|\mathbf{y}^{(2)} - \mathbf{y}^{(1)}| - 2a), \quad \mathbf{q}\cdot\mathbf{n} > \mathbf{0}, \quad [C15]$$

where $\mathbf{q} = \mathbf{w}^{(2)} - \mathbf{w}^{(1)}$, and $e$ is the coefficient of restitution. Substituting this into [C14], one finds

$$\epsilon_D\boldsymbol{\sigma}_c = -2(1 + e)a^3m\int_{\mathbf{q}\cdot\mathbf{n}>0} P(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, t)(\mathbf{q}\cdot\mathbf{n})^2\mathbf{nn}\,\mathrm{d}^3w^{(1)}\,\mathrm{d}^3w^{(2)}. \quad [C16]$$

This is identical to [3.6] in Lun *et al.* (1984) or [2.19] in Savage and Jeffrey (1981).

Since the individual torques resulting from the contact of two particles are not necessarily equal and opposite, in general the average torque cannot be written as a divergence. Indeed, in the case of rapid granular flow, a source term has to be included in the angular momentum equation (Lun 1991).